1. The intent of part (b) of this exercise is to find a rigorous solution to Exercise 4.15 of McCullagh and Nelder (1989). (See also the heuristic discussion in Cox and Snell, 1989, p. 32.)

Suppose that $X \sim \text{Bin}(n, p)$, $0 < p < 1$, and let $\lambda = \log\{p/(1 - p)\}$ be the log-odds, so that $p = e^\lambda / (1 + e^\lambda)$. For $c_1, c_2 > 0$, consider estimators of $\lambda$ of the form

$$\hat{\lambda} = \log\left(\frac{X + c_1}{n - X + c_2}\right).$$

It is not difficult to show that $\hat{\lambda}$ is equivariant under the two-element group of transformations \{\(X \mapsto X, X \mapsto n - X\)\} (see Casella and Berger, 2002, Section 6.4) if and only if $c_1 = c_2$, thus reducing our attention to estimators of the form

$$\hat{\lambda} = \log\left(\frac{X + c}{n - X + c}\right), \quad c > 0.$$

Let $c > 0$ be fixed.

(a) Derive the asymptotic distribution of $\hat{\lambda}$. (More specifically, find the limiting distribution of $\sqrt{n}(\hat{\lambda} - \lambda)$.)

(b) Show that

$$E(\hat{\lambda}) = \lambda + \frac{(1 - 2p)(2c - 1)}{2np(1 - p)} + O(n^{-2}). \quad (*)$$

This shows that choosing $c = 1/2$ reduces the order of the bias from $n^{-1}$ to $n^{-2}$.

Hint: One way to proceed in part (b) is by showing that

$$E\{\log(X + c)\} = \log(np) + \frac{c}{np} - \frac{1 - p}{2np} + O(n^{-2}). \quad (†)$$

For this, consider a three-term Taylor expansion of $\log\{(X + c)/(np)\} = \log(1 + T)$ about $T = 0$. For a rigorous proof, you will have to show that the expectation of the remainder term is $O(n^{-2})$ (Hoeffing’s inequality may be useful here). Next, find the corresponding expansion for $E\{\log(n - X + c)\}$ (just apply $$(†)$$ to $n - X \sim \text{Bin}(n, 1 - p)$ for this), and combine these two results to deduce $$(*)$$. . . Or maybe ignore this hint and try to find a simpler solution.