1. Let $X_1, \ldots, X_n$ be i.i.d. with common density

$$f(x) = \begin{cases} 
\frac{1}{10}, & 0 \leq x \leq 5, \\
\frac{1}{4}, & 5 \leq x \leq 7,
\end{cases}$$

and let $M_n$ be the sample median. Proceeding from first principles (i.e., do not just quote a result from some place), obtain the asymptotic distribution of $M_n$.

*Hint:* Take $M_n = X_{\lfloor a_n \rfloor}$, where $a_n = \lfloor \frac{n+1}{2} \rfloor$. The (unique) median of $f$ is 5, so it makes sense to consider the distribution of $\sqrt{n}(M_n - 5)$. Note that

$$P\left(n^{1/2}(M_n - 5) \leq x\right) = P\left(X_{a_n} \leq 5 + n^{-1/2}x\right) = P\left(n\hat{F}_n(5 + n^{-1/2}x) \geq a_n\right),$$

where $\hat{F}_n$ is the empirical c.d.f. Note that $n\hat{F}_n(5 + n^{-1/2}x)$ has a binomial distribution on $n$ trials with success probability $p_n$ depending on $n$ and $x$. You may find it useful to employ the result of problem 3 from assignment 3. It may also be helpful to note that

$$\frac{n-1}{2} < a_n \leq \frac{n+1}{2} \implies \left|a_n - \frac{n}{2}\right| \leq \frac{1}{2} \implies n^{1/2}\left|\frac{a_n}{n} - \frac{1}{2}\right| \leq \frac{1}{2}n^{-1/2} \to 0 \quad \text{as } n \to \infty.$$

2. Suppose that $X_1, \ldots, X_n$ is an i.i.d. sample from a univariate distribution $F$, and that we wish to estimate $\theta = \mu^3$, where $\mu$ is the mean of $F$. Give the $U$-statistic estimator of $\theta$ and find its variance and the asymptotic distribution of $\sqrt{n}(U - \theta)$. 