1. Suppose that $E(Y|X) = X$ and $E(Y^2) = E(X^2) < \infty$. Show that $Y = X$ a.s.

2. Suppose that $E(|X|) < \infty$ and $E(|Y|) < \infty$ and that $E(Y|X) = X$ and $E(X|Y) = Y$. Show $X = Y$ a.s.

3. Give an example on $\Omega = \{a, b, c\}$ in which

$$E[E(X|\mathcal{G}_1)|\mathcal{G}_2] \neq E[E(X|\mathcal{G}_2)|\mathcal{G}_1].$$

4. Assuming throughout that $E(|X|)$, $E(|Y|)$, and $E(|XY|)$ are finite, show that each of the following implications holds (in showing the second implication, do not assume independence):

$$X \perp Y \implies E(Y|X) = E(Y) \implies E(XY) = E(X)E(Y).$$

For each of the implications above, give an example of random variables $X, Y \in \{-1, 0, 1\}$ a.s. showing that the reverse implications is false.

5. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and suppose that $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ are sub-$\sigma$-fields of $\mathcal{F}$. Let $\mathcal{G}_i \vee \mathcal{G}_j := \sigma(\mathcal{G}_i, \mathcal{G}_j)$ represent the smallest $\sigma$-field containing both $\mathcal{G}_i$ and $\mathcal{G}_j$. Show that if the random variable $X$ is integrable and $\mathcal{G}_1$-measurable and the $\sigma$-fields $\mathcal{G}_1 \vee \mathcal{G}_2$ and $\mathcal{G}_3$ are independent, then $E(X|\mathcal{G}_2 \vee \mathcal{G}_3) = E(X|\mathcal{G}_2)$.

State a version of this result for the special case of the conditional expectation of a random variable $X$ given random vectors $Y$ and $Z$.

**Hint:** Show that $E(X|\mathcal{G}_2)$ is a version of $E(X|\mathcal{G}_2 \vee \mathcal{G}_3)$. In doing so, consider sets of the form $A = B \cap C$, $B \in \mathcal{G}_2$, $C \in \mathcal{G}_3$, and note that this class of sets forms a $\pi$-system.