1. Let $X_n \geq 0$ be independent for $n \geq 1$. The following are equivalent:

(i) $\sum_{n=1}^{\infty} X_n < \infty$ a.s.

(ii) $\sum_{n=1}^{\infty} \{P(X_n > 1) + E[X_n I[X_n \leq 1]]\} < \infty$.

(iii) $\sum_{n=1}^{\infty} E[X_n/(1 + X_n)] < \infty$.

2. Suppose that $\sum_{n=1}^{\infty} \sigma_n^2 n^2 = \infty$ and without loss of generality that $\sigma_n^2 \leq n^2$ for all $n \geq 1$. Show that there exist independent random variables $X_n, n \geq 1$, with $E[X_n] = 0$ and $\text{Var}(X_n) \leq \sigma_n^2$, for which $X_n/n$ does not converge to 0 a.s. Conclude from this that $\sum_{n=1}^{\infty} X_n/n$ does not converge a.s. and that $n^{-1} \sum_{i=1}^{n} X_i$ does not converge a.s. (to any random variable, let alone to the constant 0).

If $X_n, n \geq 1$, are independent with $E[X_n] = 0$ and $\text{Var}(X_n) = \sigma_n^2$, and if $\sum_{n=1}^{\infty} \sigma_n^2/n^2 < \infty$, then by Theorem 6.3.3 in the notes (or by a simple application of Kolmogorov’s convergence criterion followed by Kronecker’s lemma), $\sum_{n=1}^{\infty} X_n/n$ converges a.s. and $n^{-1} \sum_{i=1}^{n} X_i \to 0$ a.s. So the point of this problem is to show that this result can easily fail when $\sum_{n=1}^{\infty} \sigma_n^2/n^2 = \infty$.

3. Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables with a symmetric distribution, i.e., $X$ and $-X$ have the same distribution. Then $\sum_{n=1}^{\infty} (X_n/n)$ converges almost surely iff $E[|X_1|] < \infty$.

4. Let $\{X_n, n \geq 1\}$ be nondegenerate i.i.d. $L_2$ random variables and let $\sigma^2 = \text{Var} X_1$,

$$\bar{X}_n = \frac{1}{n} \sum_{j=1}^{n} X_j, \quad S_n^2 = \frac{1}{n} \sum_{j=1}^{n} (X_j - \bar{X}_n)^2, \quad n \geq 2.$$ 

Prove that $S_n \to \sigma$ almost surely and in $L^2$, and that $E S_n \to \sigma$.

5. The $L^2$ weak law generalizes immediately to certain dependent sequences. Suppose $E[X_n] = 0$ and $E[X_n X_m] \leq r(n - m)$ for $m \leq n$, where $r(k) \to 0$ as $k \to \infty$. Show that $(X_1 + \cdots + X_n)/n \to 0$ as $n \to \infty$. 