1. Let $X_1, X_2, \ldots$ be identically distributed random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$ and suppose that $E[|X_1|^p] < \infty$ for some $p > 0$.

(a) Show that $nP\left(|X_1| \geq \epsilon n^{1/p}\right) \to 0$ as $n \to \infty$, and then use this fact to show that

$$n^{-1/p} \max_{1 \leq j \leq n} |X_j| \xrightarrow{P} 0.$$  

(This is a slight generalization of Billingsley’s Exercise 21.21.)

(b) Now prove the stronger result that $n^{-1/p} \max_{1 \leq j \leq n} |X_j| \to 0$ a.s.

2. Do there exist random variables $\{X_n, n \geq 1\}$ for which the expected value of $\lim \inf_{n \to \infty} X_n$ does not exist but such that the negative parts $\{X_n^-, n \geq 1\}$ are uniformly integrable? Prove your answer.

3. Let $\{X_n, n \geq 1\}$ be a sequence of random variables, each having finite mean. Assume that $X_n \xrightarrow{P} 0$ as $n \to \infty$ and that $\sup_n \text{Var}(X_n) < \infty$. Show that $\lim_{n \to \infty} E[|X_n|] = 0$.

4. Let $\{X_n\}$ be independent random variables with common mean 0 and variance 1. Then for every $x$:

$$P\left(\max_{1 \leq k \leq n} S_k \geq x\right) \leq 2P\left(S_n \geq x - \sqrt{2n}\right)$$

Hint: Let $A_k = [S_k \geq x, S_j < x, 1 \leq j \leq k - 1]$, and show that

$$P\left(S_n \geq x - \sqrt{2n}\right) \geq \sum_{k=1}^{n} P\left(A_k \cap [S_n - S_k \geq -\sqrt{2n}]\right).$$