1. Let $X_1, X_2, \ldots$ be identically distributed random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$ and suppose that $E[|X_1|^p] < \infty$ for some $p > 0$.

   (a) Show that $nP(|X_1| \geq \epsilon n^{1/p}) \to 0$ as $n \to \infty$, and then use this fact to show that
   
   $$n^{-1/p} \max_{1 \leq j \leq n} |X_j| \overset{P}{\to} 0.$$  
   
   (This is a slight generalization of Billingsley’s Exercise 21.21.)

   (b) Now prove the stronger result that $n^{-1/p} \max_{1 \leq j \leq n} |X_j| \to 0$ a.s.

2. Let $\{X_n\}$ be independent and identically distributed with mean 0 and variance 1. Then we have for every $x$:

   $$P \left( \max_{1 \leq k \leq n} S_k \geq x \right) \leq 2P \left( S_n \geq x - \sqrt{2n} \right)$$

   Hint: Let $A_k = [S_k \geq x, S_j < x, 1 \leq j \leq k - 1]$, and show that

   $$P \left( S_n \geq x - \sqrt{2n} \right) \geq \sum_{k=1}^{n} P \left( A_k \cap \left[ S_n - S_k \geq -\sqrt{2n} \right] \right).$$