Turn in the first two problems. The remaining problems are optional problems that you may wish to try, but they will not be collected.

Note that problem 3 generalizes the “crystal ball” condition for uniform integrability.

1. Supposing that \( \{X_t : t \in T\} \) and \( \{Y_t : t \in T\} \) are uniformly integrable, prove that \( \{X_t + Y_t : t \in T\} \) is uniformly integrable.

2. Let \( \{X_n, n \geq 1\} \) be a sequence of random variables, each having finite mean. Assume that \( X_n \overset{P}{\to} 0 \) as \( n \to \infty \) and that \( \sup_n \text{Var}(X_n) < \infty \). Show that \( \lim_{n \to \infty} E[|X_n|] = 0 \). *Hint: First use Chebyshev’s inequality to show that \( \sup_n E[X_n] < \infty \).*

3. Let \( f : [0, \infty) \to [0, \infty) \) satisfy
\[
\frac{f(x)}{x} \to \infty \quad \text{as} \quad x \to \infty,
\]
and suppose that \( \{X_n, n \geq 1\} \) is a sequence of random variables satisfying
\[
\sup_{n \geq 1} E[f(|X_n|)] < \infty.
\]
Show that \( \{X_n, n \geq 1\} \) is u.i.

4. Do there exist random variables \( \{X_n, n \geq 1\} \) for which the expected value of \( \liminf_{n \to \infty} X_n \) does not exist but such that the negative parts \( \{X_n^-, n \geq 1\} \) are uniformly integrable? Prove your answer.