and it follows from the convergence half of the Borel-Cantelli lemma (4.1.2) that \( \mu(B) = 0 \), where
\[
B = \{ \omega : |f_{n+1}(\omega) - f_n(\omega)| > 2^{-j} \text{ i.o.}(j) \}.
\]
If \( \omega \in B^c \), then there exists a \( k = k(\omega) \) such that \( |f_{n+1}(\omega) - f_n(\omega)| \leq 2^{-j} \) for all \( j \geq k \), and so for all \( i \geq j \geq k \),
\[
|f_n(\omega) - f_n(\omega)| \leq \sum_{i=j}^{i-1} |f_{n+1}(\omega) - f_n(\omega)| \leq \sum_{i=j}^{\infty} 2^{-l} = 2^{-j+1} \to 0 \quad \text{as} \quad j \to \infty.
\]
This implies that for \( \omega \in B^c \), \( \{f_n(\omega) : j \geq 1\} \) is a Cauchy sequence of real numbers and hence has a finite limit (because \( \mathbb{R} \) is complete).

Let \( f(\omega) = \lim_{j \to \infty} f_n(\omega)I_{B^c}(\omega) \), i.e., \( f(\omega) = \lim_{j \to \infty} f_n(\omega) \) for \( \omega \in B^c \) and \( f(\omega) = 0 \) for \( \omega \in B \). Then \( f \) is measurable, and we claim that \( f_n \stackrel{\mu}{\to} f \). \(^1\) To see this, let
\[
A_j = \{ \omega : |f_{n+1}(\omega) - f_n(\omega)| > 2^{-j} \} \quad \text{and} \quad B_j = \bigcup_{i=j}^{\infty} A_i,
\]
so that
\[
B = \limsup_j A_j = \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} A_i = \bigcap_{j=1}^{\infty} B_j \quad \text{and} \quad B^c = \liminf_j A_j^c = \bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} A_i^c = \bigcup_{j=1}^{\infty} B_j^c.
\]
Note that
\[
\mu(B_j) \leq \sum_{i=j}^{\infty} \mu(A_i) \leq \sum_{i=j}^{\infty} 2^{-i} = 2^{-j+1}.
\]

Suppose now that \( \omega \in B_j^c \). Then for \( l \geq j \),
\[
|f_n(\omega) - f(\omega)| \leq \sum_{i=j}^{l-1} |f_{n+1}(\omega) - f_n(\omega)| + |f_n(\omega) - f(\omega)|,
\]
and because the left-hand side of this inequality does not depend on \( l \), it follows that
\[
|f_n(\omega) - f(\omega)| \leq \lim_{l \to \infty} \left( \sum_{i=j}^{l-1} |f_{n+1}(\omega) - f_n(\omega)| + |f_n(\omega) - f(\omega)| \right) = \sum_{i=j}^{\infty} |f_{n+1}(\omega) - f_n(\omega)| \leq \sum_{i=j}^{\infty} 2^{-i} = 2^{-j+1}.
\]
(Here we have used the fact that \( \omega \in B_j^c \implies \omega \in B^c \implies f_n(\omega) \to f(\omega) \) as \( l \to \infty \).)

Thus for any \( \epsilon > 0 \) and for \( j \) sufficiently large that \( 2^{-j+1} < \epsilon \), we have
\[
\mu(|f_n - f| > \epsilon) \leq \mu(|f_n - f| > 2^{-j+1}) \leq \mu(B_j) \leq 2^{-j+1} \to 0 \quad \text{as} \quad j \to \infty.
\]
This proves that \( f_n \stackrel{\mu}{\to} f \), as claimed.

\(^1\) Note that \( f_n(\omega) \to f(\omega) \) for all \( \omega \in B^c \), and \( \mu(B) = 0 \), so that \( f_n \to f \) a.e. If \( \mu \) is a finite measure, this is enough to imply \( f_n \stackrel{\mu}{\to} f \), but as we have seen, this implication fails for infinite measures.