Note: In exercise doing exercise 18.4 from Billingsley, take the σ-field on the line to be the Borel sets for both Lebesgue measure and counting measure. Then the product σ-field is $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, and because the diagonal $E = \{(x, x) : x \in \mathbb{R}\}$ is closed, $E \in \mathbb{R} \times \mathbb{R}$. Now finish the problem.

1. Let $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ be a collection of probability measures defined on a common measurable space $(\Omega, \mathcal{F})$.

   (a) Show that if $\Theta$ is countable, then there exists a probability measure $Q$ on $(\Omega, \mathcal{F})$ such that $P_\theta \ll Q$ for all $\theta \in \Theta$.

   (b) Show that if there exists a σ-finite measure $\mu$ on $(\Omega, \mathcal{F})$ such that $P_\theta \ll \mu$ for all $\theta \in \Theta$, then there exists a probability measure $Q$ on $(\Omega, \mathcal{F})$ such that $P_\theta \ll Q$ for all $P_\theta \in \Theta$.

   (c) Show by example that there may be no such dominating σ-finite dominating measure and hence no dominating probability measure.