I. Know the following definitions:
   A. Field.
   B. \( \sigma \)-field.
   C. Measure, including
      1. \( \sigma \)-finite measure.
      2. Probability measure.
   D. \( \pi \)-system.
   E. \( \lambda \)-system.
   F. Semiring.
      1. The primary example is the class \( \mathcal{I}_0 \). Knowing this example helps to remember the definition of a semiring.
      2. The other example that we worked with is \( \mathcal{I}_0^k \).
   G. Borel \( \sigma \)-field on \( \mathbb{R} \) and \( \mathbb{R}^k \).
   H. Borel \( \sigma \)-field on \( \mathbb{R}^k \).
II. Be familiar with the examples and counterexamples given in the notes and the exercises.
III. Know the definition and basic theorems concerning \( \sigma(\mathcal{A}) \), the \( \sigma \)-field generated by the class \( \mathcal{A} \).
   A. Note that the proof of part (iii) of theorem 1.1.5 is our first encounter with arguing via the "good sets principle."
IV. You should be able to show that the Borel sets are generated by any of the various classes mentioned in the notes.
V. Know the fundamental properties of measures.
   A. The proofs of most of these are instructive (with the exception of the inclusion-exclusion formula).
VI. Uniqueness results.
   A. Be familiar with the conclusion of the \( \pi-\lambda \) theorem, but you will not be asked to prove it.
   B. You should be able to do the proof of the first uniqueness theorem for finite measures and the lemma leading up to it. Don’t worry about the proofs of the other versions of the uniqueness theorem, but you should be able to use them.
VII. Extension.

A. Outer measure.
   1. Know the general definition of an outer measure.
   2. The proof of lemma 1.3.8.
   3. Definition of $\mu^*$-measurability.
   4. Know the statement of the Carathéodory extension theorem.
   5. You do not have to know the proof of the extension theorem, but you should
      know in broad outline how the theorem comes about. Specifically:
      a. How is the outer measure $\mu^*$ defined.
      b. What are the roles of $\mu^*$ and $\mathcal{M}(\mu^*)$ in the proof.

VIII. Completion.

A. Know the definition of a complete measure space.
B. Know the content and be able to prove the basic theorem 1.3.14 on completion,
   and of course exercise 1.3.1.
C. Know theorem 1.3.17 and be able to prove it.
D. Know theorem 1.3.20, but do not worry about its proof.

IX. Lebesgue-Stieltjes Measures on $\mathbb{R}$.

A. Know the definition of a L-S measure.
B. Know the statement of the correspondence theorem, but you do not have to be
   able to prove it.
   1. The proof of one direction (given an L-S measure there is a corresponding
      generalized distribution function) is easy.
   2. The proof of the other direction (the useful one, that given a gdf there is
      a unique corresponding L-S measure) is harder, but you should be familiar
      with the broad outline:
      a. Define $\mu$ on the semiring $\mathcal{I}_0$ (how?).
      b. Show that $\mu$ satisfies the conditions of the Carathéodory extension theo-
         rem on $\mathcal{I}_0$. This implies the existence of an extension of $\mu$ to
         $\mathcal{B} = \sigma(\mathcal{I}_0)$.
      c. Uniqueness of the extension then follows because $\mu$ is $\sigma$-finite on
         $\mathcal{I}_0$.

X. Lebesgue-Stieltjes Measures on $\mathbb{R}^k$.

A. Know how $\mathbb{R}^k$ is defined.
B. Know the definition of a $k$-dimensional (generalized) distribution function and
   the basic result of the $k$-dimensional version of the correspondence theorem, but
   don’t worry about any details.
XI. Lebesgue Measure.
   A. You should be able to prove translation invariance of Lebesgue measure \( \lambda \). It might be helpful to work through the one dimensional case, which makes the notation simpler.
   B. You should have some familiarity with the outline of the construction of a non-measurable set, but you will not be asked any details about the construction.

XII. Measurability.
   A. Know the definition of a measurable mapping (function) and of a random variable.
   B. Be able to to prove the basic theorems concerning measurability:
      1. a composition of measurable maps is measurable.
      2. measurability is determined by any generating class in the range space.
   C. For real and extended-real-valued functions:
      1. Know how to verify measurability.
      2. Know the basic results for constructing new measurable functions from old.
         I think it’s probably best to study the second proof given in the hand-written notes of things like “\( f, g \) measurable implies \( f + g \) measurable”.
      3. Prove that \( [f < g] \in \mathcal{F} \) for measurable \( f \) and \( g \).
      4. Be able to prove that the limit of a sequence of measurable functions is measurable, and related results.
      5. A nonnegative measurable function can be written as the nondecreasing limit of a sequence of nonnegative, measurable simple functions. You will not be asked to prove this fact, but you might have to use it.

XIII. Induced measures and distributions.
   A. Be familiar with the notion of the measure induced by a measurable mapping and the special case of the distribution of a random variable.
   B. Be familiar with the two constructions of a probability space and a random variable having a given distribution function \( F \).

XIV. Exercises. Most of the assigned exercises are doable. In particular, you shouldn’t be too surprised if one of the following showed up on the exam:
   A. 2.3, 2.4, 2.9, 2.11(a,c);
   B. 2.13, 2.17, 3.11, 10.1, 10.2;
   C. 2.19(a,b), 3.3(a–d,f), 3.12;
   D. 12.1(one-dimensional case only), 13.8.
   E. From the list of additional exercises: 2.5, 10.3, 13.1, 13.6, 13.11, 13.14.