1. If $Y$ represents the cosine of the angle at which an electron is emitted in muon decay, then $Y$ has density

$$f(y) = \begin{cases} \frac{1}{2}(1 + \theta y), & -1 \leq y \leq 1, \\ 0, & \text{otherwise}, \end{cases}$$

where $-1 \leq \theta \leq 1$. Suppose that $Y_1, \ldots, Y_n$ are a random sample of such cosines.

(a) Find the method of moments estimator of $\theta$. \hspace{1cm} (5 pts)

$$\mu'_1 = E(Y) = \int_{-1}^{1} y \frac{1}{2}(1 + \theta y) \, dy = \frac{1}{2} \left[ \int_{-1}^{1} (y + \theta y^2) \, dy = \frac{1}{2} \left( \frac{1}{2} y^2 + \frac{\theta}{3} y^3 \right) \right]|_{-1}^{1} = \frac{\theta}{3}$$

$$m'_1 = \bar{Y}$$

$$\mu'_1 = m'_1 \implies \frac{\theta}{3} = \bar{Y} \implies \theta = 3\bar{Y}$$

Thus the method of moments estimator is

$$\hat{\theta}_{MME} = 3\bar{Y}$$

(b) Find the equation that must be solved in order to find the MLE of $\theta$. \textit{Note: You must complete any derivatives, etc., needed to set up the likelihood equation, but do NOT attempt to solve the equation}. \hspace{1cm} (5 pts)

$$L = \prod_{i=1}^{n} f(y_i) = \prod_{i=1}^{n} \left[ \frac{1}{2}(1 + \theta y_i) \right] = 2^{-n} \prod_{i=1}^{n} (1 + \theta y_i)$$

$$\ln(L) = \sum_{i=1}^{n} \ln(1 + \theta y_i) - n \ln(2)$$

$$\frac{d}{d\theta} \ln(L) = \sum_{i=1}^{n} \frac{1}{1 + \theta y_i} \frac{d}{d\theta} (1 + \theta y_i) = \sum_{i=1}^{n} \frac{y_i}{1 + \theta y_i}$$

Thus the likelihood equation is

$$\sum_{i=1}^{n} \frac{y_i}{1 + \theta y_i} = 0$$