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Source: *Biometrics*, Vol. 36, No. 1 (Mar., 1980), pp. 59-67
Published by: International Biometric Society
Stable URL: http://www.jstor.org/stable/2530495
Accessed: 20-12-2016 15:49 UTC

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Generalized Odds Ratios for Ordinal Data

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SUMMARY

We consider properties of the ordinal measure of association defined by the ratio of the proportions of concordant and discordant pairs. For 2×2 cross-classification tables, the measure simplifies to the odds ratio. The generalized measure can be used to summarize the difference between two stochastically ordered distributions of an ordinal categorical variable. The ratio of its values for two groups constitutes an odds ratio defined in terms of pairs of observations. Unlike the odds ratio measures proposed by Clayton (1974, *Biometrika* **61,** 525–531) for similar purposes, this measure is not linked to specific model assumptions and hence it is more widely applicable as a descriptive measure.

1. Introduction

In many biological and medical applications, response variables are measured on categorical scales having ordered, though non-numerically valued, levels. Clayton (1974) has proposed some statistics based on odds ratios, which are designed to summarize the difference in location between two distributions of an ordinal categorical variable. The statistics correspond to a logit model for the sets of cumulative proportions of the two distributions. He also has proposed generalizations of these statistics to describe association between two ordinal categorical variables.

We consider a simpler generalization for ordinal data of the odds ratio, one that has wider applicability as a summary measure. For the bivariate ordinal case, this measure equals the ratio of the proportions of concordant and discordant pairs. For comparing the distributions of two random variables, Y_1 and Y_2, it estimates $\mathrm{pr}\,(Y_2 > Y_1)/\mathrm{pr}\,(Y_1 > Y_2)$. The ratio of its values for two groups constitutes an odds ratio defined in terms of pairs of observations. These statistics have simple interpretations and are also well-defined for continuous variables. They can be used to summarize the extent to which bivariate relationships are monotonic or the extent to which one ordinal distribution tends to exceed another one. Unlike Clayton, whose inferential emphasis was on testing hypotheses (and hence on stating approximate null distributions) for a particular model, we concentrate on interval estimation (and hence on stating approximate non-null distributions) for these measures.

2. Clayton's Statistics

Consider first the comparison of two distributions of an ordinal categorical variable. For the ith distribution, let $\tilde{\rho}_{ij}$ be the conditional probability concentrated in the jth category of the ordinal variable, $j = 1, 2, \ldots, c$. Let

$$F_{1k} = \sum_{j=1}^{k} \tilde{\rho}_{1j}, \qquad F_{2k} = \sum_{j=1}^{k} \tilde{\rho}_{2j}, \qquad k = 1, \ldots, c,$$

Key words: Concordance; Odds ratio; Ordinal measure of association; Gamma; Kendall's tau.

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represent the two distribution functions. Clayton's statistics refer to the model

$$\frac{F_{1k}}{1-F_{1k}} = \theta \frac{F_{2k}}{1-F_{2k}}, \qquad k = 1, \ldots, c-1. \tag{2.1}$$

That is, Clayton assumed equality of the $c-1$ odds ratios $\{F_{1k}(1-F_{2k})/F_{2k}(1-F_{1k})$, $k = 1, \ldots, c-1\}$ obtained for the 2×2 tables corresponding to all possible dichotomizations of the dependent variable. Notice that $\log \theta$ is the difference between the distributions on a logistic scale. For this model, Clayton considered estimation of θ and $\log \theta$ based on independent random samples from the two distributions. His estimators of $\log \theta$ are weighted averages of the $c-1$ log sample odds ratios and pooled estimators, the weights being chosen to minimize the asymptotic variances of the estimators in the neighbourhood of $\theta = 1$. Clayton (1976) also generalized these estimators for the case in which some observations are subject to censorship. McCullagh (1977) proposed a related model for which $\log \theta$ is estimated using paired comparisons on the ordinal variable.

The parameter θ is a simple measure for comparing the two distributions if model (2.1) is at least approximately accurate. However, the fit provided by this model can be very crude even if the categorical distributions are based on underlying distributions differing only by a shift in location. Fleiss (1970) showed that if two groups have the distributions $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$, and if several 2×2 tables are formed by dichotomizing the combined populations at various points, then the odds ratio is very unstable compared to other measures for 2×2 tables. For example, if $\mu_2 - \mu_1 = \sigma$, the value of the odds ratio ranges between 5.02 and 10.16 as the dichotomization varies from $(0.50, 0.50)$ to $(0.95, 0.05)$ for the combined populations.

Clayton (1974) generalized his estimates of θ to the case of two ordinal categorical variables, the joint distribution of which forms a $r \times c$ cross-classification table. The generalized statistics are based on a model which assumes the existence of a common odds ratio θ for all $(r-1)(c-1)$ possible ways of collapsing the table into a 2×2 table. Clayton developed estimates of $\log \theta$, such as weighted averages of the $(r-1)(c-1)$ sample log odds ratios. The constant odds ratio assumption is a natural one for a bivariate model in which this statistic is stable, such as for the family of bivariate distributions proposed by Plackett (1965). However, it is badly violated for many other bivariate distributions. For the case of an underlying bivariate normal distribution, for example, Mosteller (1968) showed that the value of the odds ratio is highly dependent on the points of dichotomization for the 2×2 table unless the correlation is quite close to zero. When $\rho = 0.75$, the odds ratio varies between 11.2 and 200.8 for the dichotomizations considered by Mosteller. As another illustration, Table 2 lists the values of the odds ratio for all possible 2×2 condensations of the 4×4 cross classification describing quality of left and right eyesight given for men and for women in Table 1. The odds ratio varies from 4.75 to 44.05 for women and from 6.43 to 30.00 for men.

3. Another Generalization of the Odds Ratio

We now suggest an alternative generalization of the odds ratio for the comparison of two groups on an ordinal scale and for the measurement of ordinal association. It is similar in nature to Clayton's generalizations in the sense that it is a single summary measure which simplifies to the odds ratio in the 2×2 case. However, it is proposed in a different spirit since it is not linked to a specific model and hence it does not assume a constant odds ratio for all 2×2 condensations of the table. In this respect it is similar to measures of association proposed by Goodman and Kruskal (1954). We will define our proposed

Table 1
Unaided distance vision; from Stuart (1953)

Sex	Grade of right eye	Grade of left eye			
		Highest	Second	Third	Lowest
Women	Highest	1520	266	124	66
	Second	234	1512	432	78
	Third	117	362	1772	205
	Lowest	36	82	179	492
Men	Highest	821	112	85	35
	Second	116	494	145	27
	Third	72	151	583	87
	Lowest	43	34	106	331

generalization for the case of a cross-classification table for ordinal variables and then consider its specialized use for comparing two distributions of an ordinal categorical variable.

Consider the $r \times c$ table of probabilities representing the joint distribution of two ordinal categorical variables. Let P_c denote the probability that a randomly selected pair of members is concordant and let P_d denote the probability of discordance. The ratio

$$\alpha = P_c/P_d \tag{3.1}$$

is an easily interpretable ordinal measure of association which describes the extent to which there is a monotonic increasing or decreasing relationship. Letting ρ_{ij} denote the probability that a member is classified in row i and column j, we have

$$\alpha = \sum_{i,j} \rho_{ij} \mathscr{R}_{ij}^{(s)} \Big/ \sum_{i,j} \rho_{ij} \mathscr{R}_{ij}^{(d)}, \tag{3.2}$$

where

$$\mathscr{R}_{ij}^{(s)} = \sum_{i'>i} \sum_{j'>j} \rho_{i'j'} + \sum_{i'<i} \sum_{j'<j} \rho_{i'j'}$$

and

$$\mathscr{R}_{ij}^{(d)} = \sum_{i'>i} \sum_{j'<j} \rho_{i'j'} + \sum_{i'<i} \sum_{j'>j} \rho_{i'j'}.$$

For the special case of a 2×2 table, $P_c = 2\rho_{11}\rho_{22}$ and $P_d = 2\rho_{12}\rho_{21}$, so that $\alpha = \rho_{11}\rho_{22}/\rho_{12}\rho_{21}$. Thus α, which is defined to be the odds of selecting a concordant pair relative to selecting a discordant pair, is a generalization of the odds ratio. Clearly, $0 \leq \alpha \leq \infty$ with $\alpha = 1$ if, but not only if, the variables are independent. By analogy with the log odds ratio that is often used with 2×2 tables, $\log \alpha$ is a useful related measure. It is symmetric around the independence value of zero, in the sense that a reversal of the positions of P_c and P_d (such as occurs in reversing the order of the levels of one of the variables) results in a change in its sign. The magnitude of $\log \alpha$ is not as easily interpretable as that of α, but the distribution of its sample version tends to be more symmetric and to converge to normality faster than the distribution of the sample version of α. Neither α nor $\log \alpha$ makes a distinction between response and explanatory variables.

Several measures of association have been formulated to generalize Kendall's tau for use with cross-classification tables having ordered rows and ordered columns. One of the most commonly used of these is $\gamma = (P_c - P_d)/(P_c + P_d)$, proposed by Goodman and

Kruskal (1954). The quantity α is simply the monotonic transformation of gamma: $\alpha = (1 + \gamma)/(1 - \gamma)$.

A random sample version of α is given by

$$\hat{\alpha} = \hat{P}_c/\hat{P}_d = \sum_{i,j} p_{ij} \hat{\mathcal{R}}_{ij}^{(s)} \bigg/ \sum_{i,j} p_{ij} \hat{\mathcal{R}}_{ij}^{(d)}, \tag{3.3}$$

in which the $\{\rho_{ij}\}$ in α are replaced by the sample proportions $\{p_{ij}\}$. When $0 < \alpha < \infty$, the asymptotic distribution of $\hat{\alpha}$ under random sampling may be easily obtained using the 'delta method' outlined by Goodman and Kruskal (1972). Namely, $(\hat{\alpha} - \alpha)/\hat{\sigma}$ tends in distribution to N(0, 1) as $n \to \infty$, where

$$n\hat{\sigma}^2 = 4 \left\{ \sum_{i,j} p_{ij} (\hat{\alpha}\hat{\mathcal{R}}_{ij}^{(d)} - \hat{\mathcal{R}}_{ij}^{(s)})^2 \right\} \bigg/ \hat{P}_d^2. \tag{3.4}$$

The z statistic based on $\hat{\alpha}$ for testing independence is asymptotically equivalent to a z statistic based on Kendall's tau. To see this, note that $(\hat{\alpha} - 1)\sqrt{n} = (\hat{P}_c - \hat{P}_d)\sqrt{n}/\hat{P}_d$, which has the same asymptotic distribution as does $(\hat{P}_c - \hat{P}_d)\sqrt{n}/P_d$. Thus, $z = (\hat{\alpha} - 1)/\sigma$ is asymptotically equivalent in distribution to $(\hat{P}_c - \hat{P}_d)/\sigma_{C-D}$, where $\sigma_{C-D} = P_d\sigma$. It follows from Simon (1978) that $\hat{\alpha}$ has the same efficacy as all ordinal measures of association having $P_c - P_d$ as a numerator, and thus it is locally as efficient as those measures at detecting departures from independence.

4. Comparison of Two Distributions of an Ordinal Variable

We now apply this method to the problem considered by Clayton (1974) of comparing two distributions of an ordinal categorical variable. For the case of a $2 \times c$ table, α reduces to

$$\alpha = \sum_{j > i} \rho_{1i}\rho_{2j} \bigg/ \sum_{j < i} \rho_{1i}\rho_{2j} = \sum_{j > i} \tilde{\rho}_{1i}\tilde{\rho}_{2j} \bigg/ \sum_{j < i} \tilde{\rho}_{1i}\tilde{\rho}_{2j}. \tag{4.1}$$

Let Y_1 and Y_2 be independent random variables having distributions $\{\tilde{\rho}_{1i}\}$ and $\{\tilde{\rho}_{2i}\}$, respectively. Notice that α can be expressed as

$$\alpha = \text{pr}(Y_2 > Y_1)/\text{pr}(Y_1 > Y_2). \tag{4.2}$$

If Y_2 is stochastically larger then Y_1 (i.e. $F_{2k} \leq F_{1k}$ for all k) then it is easily seen that $\alpha > 1$. Generally, α provides a summary measure of the extent to which the distribution of Y_2 falls above that of Y_1. In this case, for a random sample of size n or independent random samples of sizes n_1 and n_2 (with $n_1 + n_2 = n$), $(\hat{\alpha} - \alpha)/\hat{\sigma}$ tends in distribution to N(0, 1) as $n \to \infty$, where

$$\hat{\sigma}^2 = \left\{ (1/n_1) \sum_j \tilde{p}_{1j} \left(\hat{\alpha} \sum_{i<j} \tilde{p}_{2i} - \sum_{i>j} \tilde{p}_{2i} \right)^2 + (1/n_2) \sum_j \tilde{p}_{2j} \left(\hat{\alpha} \sum_{i>j} \tilde{p}_{1i} - \sum_{i<j} \tilde{p}_{1i} \right)^2 \right\} \bigg/ \left(\sum_{i>j} \tilde{p}_{1i}\tilde{p}_{2j} \right)^2 \tag{4.3}$$

and the $\{\tilde{p}_{ij}\}$ are the sample analogs of the $\{\tilde{\rho}_{ij}\}$. In either of the settings we have discussed, the variance of $\log \hat{\alpha}$ can be estimated for large samples by $\hat{\sigma}^2/\hat{\alpha}^2$. Thus, we can exploit the faster convergence of $\log \hat{\alpha}$ to normality by forming the large-sample $100(1 - p)\%$ confidence interval $\log \hat{\alpha} \pm Z_{p/2}\hat{\sigma}/\hat{\alpha}$ for $\log \alpha$ [where $Z_{p/2}$ is the $100(p/2)$th percentile of the standard normal distribution] and then exponentiating to obtain a corresponding confidence interval for α.

The ratio $\text{pr}(Y_2 > Y_1)/\text{pr}(Y_1 > Y_2)$ is also a simple descriptive measure for comparing two ordinal categorical distributions when matched pairs are selected from the two

distributions. Letting q_{ij} denote the proportion of matched pairs for which the observation from the first distribution falls in the ith category and the observation from the second distribution falls in the jth category, we have the analogous measure

$$\hat{\alpha}' = \left(\sum_{i<j} q_{ij}\right) \Big/ \left(\sum_{i>j} q_{ij}\right). \tag{4.4}$$

The estimated asymptotic variance of $\hat{\alpha}'\sqrt{n}$ for a random sample of n pairs is

$$\hat{\alpha}'\left(1-\sum q_{ii}\right) \Big/ \left(\sum_{i>j} q_{ij}\right)^2. \tag{4.5}$$

Of course, the sign test for paired comparisons is basically a test of whether $\alpha' = 1$.

5. Comparing and Pooling Alphas

Consider now a three-dimensional table which consists of k layers of $r \times c$ tables having ordered columns and (if $r > 2$) ordered rows. The k layers might represent k levels of a nominal or ordinal control variable, or perhaps all combinations of levels of a set of control variables. In many studies, researchers are interested in comparing the two-way associations of these k layers. Let $\alpha_1, \ldots, \alpha_k$ denote the values of α within these k layers. A simple summary comparison measure for a particular pair of layers is α_i/α_j or alternatively its logarithm. The ratio α_i/α_j is an odds ratio for pairs: namely, the odds in the ith layer of selecting a concordant pair relative to selecting a discordant pair divided by the odds in the jth layer of selecting a concordant pair relative to selecting a discordant pair. When $r = c = 2$, α_i/α_j simplifies to the ratio of the odds ratios from the two layers, the standard measure of interaction for a three-dimensional table.

If independent random samples are selected from the k layers then the variance of $\log(\hat{\alpha}_i/\hat{\alpha}_j)$ can be approximated for large samples by

$$\hat{\sigma}_{i \cdot j}^2 = (\hat{\sigma}_i^2/\hat{\alpha}_i^2) + (\hat{\sigma}_j^2/\hat{\alpha}_j^2), \tag{5.1}$$

where $\hat{\sigma}_i^2$ and $\hat{\sigma}_j^2$ are the values of $\hat{\sigma}^2$ obtained from (3.4) evaluated for the ith and jth layers. For testing a general hypothesis $H_0: \alpha_1 = \alpha_2 = \cdots = \alpha_k$ of equal association across the k layers we can use the statistic

$$\sum_{i=1}^{k} \{\hat{\alpha}_i^2 (\log \hat{\alpha}_i - \bar{L})^2/\hat{\sigma}_i^2\}, \tag{5.2}$$

where $\bar{L} = \sum_i (\hat{\alpha}_i^2 \log \hat{\alpha}_i/\hat{\sigma}_i^2)/\sum_i (\hat{\alpha}_i^2/\hat{\sigma}_i^2)$. If independent random samples are selected from the layers then, as the $n_i \to \infty$, this statistic is approximately distributed as χ^2 with $k-1$ degrees of freedom under the null hypothesis. For large samples

$$\sum_{i=1}^{k} \{(\hat{\alpha}_i - \bar{\alpha})^2/\hat{\sigma}_i^2\}, \tag{5.3}$$

where $\bar{\alpha} = \sum_i (\hat{\alpha}_i/\hat{\sigma}_i^2)/\sum_i (1/\hat{\sigma}_i^2)$, will have approximately the same distribution, though convergence may be faster with the first statistic due to the tendency for faster convergence to normality of the $\log \hat{\alpha}_i$. If $H_0: \alpha_1 = \alpha_2 = \cdots = \alpha_k$ is not rejected, one may wish to pool the $\{\log \hat{\alpha}_i\}$ in order to obtain a better estimate of the assumed common value of $\log \alpha$ or α. The measure $\bar{L}$, which approximates the weighted average of the $\{\log \hat{\alpha}_i\}$ having smallest variance under this assumption, is asymptotically normally distributed around $\log \alpha$ with estimated asymptotic variance $(\sum \hat{\alpha}_i^2/\hat{\sigma}_i^2)^{-1}$. Also, if we expect the

$\{\log \alpha_i\}$ to have the same sign we can use this distribution to test the null hypothesis $H_0 : \alpha_1 = \alpha_2 = \cdots = \alpha_k = 1$ of no partial association.

6. Alpha for Continuous Variables

If two ordinal or higher-level variables are continuous, or if one variable is continuous and the other is discrete, α is still a meaningful summary measure. When both variables are continuous, for example, $P_c + P_d = 1$, and α is related to Kendall's tau by $\alpha = (1 + \tau)/(1 - \tau)$. Since the random sample version $\hat{\tau}$ of tau is asymptotically normally distributed, it follows that $(\hat{\alpha} - \alpha)\sqrt{n}/\sqrt{\{\lim \text{var}(\hat{\alpha}\sqrt{n})\}}$ tends in distribution to N(0, 1) when $0 < \alpha < \infty$. Using expression (10.9) of Noether (1967, p.74) for $\text{var}\{n(n-1)\hat{\tau}/2\}$, we obtain

$$\lim_{n \to \infty} \text{var}(\hat{\alpha}\sqrt{n}) = \lim_{n \to \infty} \text{var}(\hat{\tau}\sqrt{n})/4(1 - P_c)^4 = 4(P_{cc} - P_c^2)/(1 - P_c)^4. \tag{6.1}$$

Here P_{cc} is the probability that for a random sample of three members, the first forms a concordant pair both when matched with the second and when matched with the third. For a bivariate normal distribution, $\alpha = (\pi + 2\sin^{-1}\rho)/(\pi - 2\sin^{-1}\rho)$, and it follows from the expression given by Kendall (1970, p. 126) for the variance of $\hat{\tau}$ in this case that

$$\lim_{n \to \infty} \text{var}(\hat{\alpha}\sqrt{n}) = \left\{\frac{1}{9} - \left(\frac{2}{\pi}\sin^{-1}\tfrac{1}{2}\rho\right)^2\right\} \bigg/ \left(\frac{\pi - 2\sin^{-1}\rho}{2\pi}\right)^4. \tag{6.2}$$

For comparing the distributions of two independent continuous random variables, $\alpha = \text{pr}(Y_2 > Y_1)/\{1 - \text{pr}(Y_2 > Y_1)\}$. Now let Y_1 and Y_1' represent independent random variables having the first distribution and let Y_2 and Y_2' represent independent random variables having the second distribution. Let $P_{21} = \text{pr}(Y_2 > Y_1)$, $P_{221} = \text{pr}(Y_1 < Y_2, Y_1 < Y_2')$, $P_{211} = \text{pr}(Y_1 < Y_2, Y_1' < Y_2)$ and suppose that random samples of sizes $n_1 = w_1 n$ and $n_2 = w_2 n$ (with $w_1 + w_2 = 1$) are selected from the two distributions. It follows from the asymptotic normality of the Mann–Whitney statistic, U = number of pairs for which $Y_2 > Y_1$, that the sample version $\hat{\alpha}$ of α is asymptotically normally distributed with

$$\lim_{n \to \infty} \text{var}(\hat{\alpha}\sqrt{n}) = \sigma_{21}^2/(1 - P_{21})^4, \tag{6.3}$$

where $\sigma_{21}^2 = \lim \text{var}(\hat{P}_{21}\sqrt{n})$. Since the sample version $\hat{P}_{21}$ of P_{21} is simply $U/n_1 n_2$, from (2.21) in Lehmann (1975, p. 70) we obtain

$$\sigma_{21}^2 = (P_{221} - P_{21}^2)/w_1 + (P_{211} - P_{21}^2)/w_2. \tag{6.4}$$

7. Examples

We conclude with two examples. The first of these illustrates the use of α for measuring association in cross classifications of ordinal variables and for summarizing paired comparisons on an ordinal response. In the second example, α is applied as a measure of the difference between two ordinal categorical distributions for independent samples.

The data in Table 1 were first presented by Stuart (1953). As shown in Table 2, the odds ratio for the nine subtables of both the cross classification for men and the cross classification for women is very unstable, sufficiently so to make Clayton's model for ordinal association inappropriate. For the 7477 women in the sample, there were 14 940 643 concordant pairs and 1 676 387 discordant pairs so that $\hat{\alpha}_1 = 8.912$. That is, there were

Table 2

Odds ratio for all condensations of cross classifications of Table 1 into 2×2 tables

Sex	Row before cut	Column before cut		
		Highest	Second	Third
Women	Highest	44.05	12.67	4.75
	Second	14.30	22.38	7.77
	Third	8.12	8.52	30.09
Men	Highest	30.00	10.93	7.42
	Second	11.72	19.50	12.09
	Third	6.43	10.42	31.31

8.912 times as many pairs of women for which the one with better right eye has better left eye than there are pairs for which the one with better right eye has poorer left eye. The estimated non-null variance of $\hat{\alpha}_1$ is, from (3.4), $\hat{\sigma}^2 = 1164.1/7477 = 0.156$ and an approximate 95% confidence interval for α_1 is $8.91 \pm 1.96\sqrt{0.156}$ or (8.14, 9.68). Similarly, $\log \hat{\alpha}_1 = 2.187$ has estimated standard error $\{1164.1/(8.912)^2 7477\}^{1/2} = 0.044$ which leads to the approximate 95% confidence interval (2.100, 2.274) for $\log \alpha$ and, exponentiating, (8.17, 9.72) for α. The two approaches yield very similar results in the confidence interval for α in this case, due to the very large sample size. For the men in the sample, $\hat{\alpha}_2 = 7.917$ and an approximate 95% confidence interval for α_2, using the asymptotic normality of $\log \hat{\alpha}_2$, is (7.02, 8.94).

We can compare the associations for the two sexes using the statistic $\hat{\alpha}_1/\hat{\alpha}_2 = 1.126$. That is, the sample ratio of concordant to discordant pairs was 1.126 times greater for women than for men. The estimated standard error of $\log(\hat{\alpha}_1/\hat{\alpha}_2) = 0.1187$ is $\{0.156/(8.91)^2 + 0.239/(7.92)^2\}^{1/2} = 0.076$ which yields an approximate 95% confidence interval of $(-0.030, 0.268)$ for $\log(\hat{\alpha}_1/\hat{\alpha}_2)$ and (0.97, 1.31) for α_1/α_2. Hence there is an insignificant difference in the values. The pooled estimate of a common value of $\log \alpha$ is $\bar{L} = 2.147$ which has an estimated standard error of 0.036.

For Table 1, one might also be interested in comparing the marginal distributions of $Y_1 =$ grade of right eye and $Y_2 =$ grade of left eye. This can be done by viewing Table 1 as the result of a sample of paired comparisons and by estimating $\alpha' = \mathrm{pr}(Y_2 > Y_1)/\mathrm{pr}(Y_1 > Y_2)$. We obtain $\hat{\alpha}' = 1.159$ for women, with $\log \hat{\alpha}' = 0.148$ having an approximate standard error of 0.043. In other words there were 1.159 times as many women having better left eye than there were having better right eye, and this difference from 1.0 attains significance at the commonly-used levels. For men, however, we obtain $\hat{\alpha}' = 0.941$, with $\log \hat{\alpha}' = -0.061$ having an approximate standard error of 0.063, so that right eye and left eye qualities are not significantly different for that sex. McCullagh (1978) proposed several models for square contingency tables having ordered categories. Our summary measures are consistent with the conclusions made using his models.

Clayton (1974) used the frequencies in Table 3 [originally presented by Holmes and Williams (1954)] relating tonsil size for two groups of children to illustrate his estimates of θ. If we consider all carrier–noncarrier pairs in that table, there are $19(560 + 269) + 29(269) = 23\,552$ pairs for which the noncarriers have larger tonsils and 39 781 pairs for which the carriers have the larger tonsils. Hence $\hat{\alpha} = 39\,781/23\,552 = 1.69$, compared to Clayton's estimates for θ of 1.78 and 1.77. The interpretation of $\hat{\alpha}$ is very simple; namely,

Table 3

Size of tonsils of carriers and noncarriers of Streptococcus pyogenes; *table analyzed in Clayton* (1974)

	Tonsils present, but not enlarged	Tonsils enlarged	Tonsils greatly enlarged
Noncarriers	497	560	269
Carriers	19	29	24

there are 1.69 times as many carrier–noncarrier pairs in the sample for which the carrier has the larger tonsils as there are pairs for which the noncarrier has the larger tonsils. From (4.3), the estimated variance of $\hat{\alpha}$ is $\hat{\sigma}^2 = 0.121$ and hence the estimated standard error of $\log \hat{\alpha}$ is $\sqrt{0.121}/1.69 = 0.206$. An approximate 95% confidence interval for $\log \alpha$ is $\log (1.69) \pm 1.96(0.206)$ or $(0.120, 0.928)$ which yields the interval $(1.13, 2.53)$ for α.

8. Conclusion

The measures presented in this paper are generalizations of the odds ratio for ordinal data. The measure α summarizes the extent to which a bivariate ordinal association is monotonic in nature and it also naturally summarizes the difference between two distributions of an ordinal variable when one is stochastically larger than the other. Of course, it is desirable to find a parsimonious model such as one assuming a constant difference in logits or a complementary log model (as developed by P. McCullagh in an unpublished report: Technical Report No. 83, Department of Statistics, University of Chicago, 1979), which can provide a reasonable fit to the data, particularly if the model is meaningful theoretically. We feel that, in the manifestations explored in this article, α complements such models even when they do fit well, particularly since it is so easily interpretable. It is easy for a researcher to understand a conclusion such as 'Considering all pairs of patients for which one received treatment A and one received treatment B, there are $\hat{\alpha} = 2.6$ times as many pairs for which treatment A results in better recovery than there are pairs for which treatment B results in better recovery'.

Although we have presented α solely as a summary descriptive measure it could also be used as the basis of a model defined for an ordinal dependent variable by utilizing pair scores. For example, let $\alpha(x_1, x_2)$ be the value of α for pairs of members having the values x_1 and x_2 on an independent variable X. We could model $\alpha(x_1, x_2) = \alpha(d)$ as depending only on the distance $d = x_1 - x_2$ between the members on X. For example, the model $\log \alpha(d) = \beta d$ is a linear logistic model for the probability that an untied pair of observations is concordant; see Schollenberger *et al.* (1979) for details. Of course, in many applications it is also important to study more specific measures such as relative risks for certain levels of the response variable.

RÉSUMÉ

Nous considérons les propriétés de la mesure ordinale d'association définie par le rapport des proportions des paires concordantes et discordantes. Pour des tableaux de classification croisée 2×2, la mesure se ramène au rapport des paris. La mesure généralisée peut être utilisée pour résumer la différence entre deux distributions stochastiques classées d'une variable catégorielle ordinale. Le rapport de ses valeurs pour deux groupes constitue un rapport de pari défini en fonction des paires d'observations. A la différence de la mesure de rapport de pari proposée par Clayton (1974, *Biometrika* **61,** 525–531) pour des objectifs similaires, cette mesure n'est pas liée à des hypothèses sur un modèle spécifique, et donc il est plus largement applicable comme une mesure descriptive.

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