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Optimal Testing in Functional Analysis of Variance Models

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PLAN

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INTRODUCTION

Analysis of variance (ANOVA) - one of the most widely used tools in applied statistics. Useful for handling low dimensional data, limitations in analyzing functional responses.

Functional analysis of variance (FANOVA) methods provide alternatives to classical ANOVA methods while still allowing a simple interpretation.


Fitting and Estimation of Components: Wahba et al., 1995; Stone et al., 1997; Huang, 1998; Lin, 2000; Gu, 2002.
**MODEL**

- Diffusion version of FANOVA. One observes a series of sample paths of a stochastic process driven by

\[ dY_i(t) = m_i(t) \, dt + \epsilon \, dW_i(t), \quad i = 1, \ldots, r; \quad t \in [0, 1]^d, \]

where \( \epsilon > 0 \) is the diffusion coefficient, \( r \) and \( d \) are finite integers, \( m_i \) are (unknown) \( d \)-dimensional response functions and \( W_i \) are independent \( d \)-dimensional standard Wiener processes.

- Results of Brown & Low (1996): Under general conditions, the corresponding discrete model is asymptotically equivalent to the diffusion model with \( \epsilon = \sigma / \sqrt{n} \).
MODEL

[Antoniadis, 1984]: Each of the \( r \) response functions in model admits the following unique decomposition

\[
m_i(t) = m_0 + \mu(t) + a_i + \gamma_i(t) \quad i = 1, \ldots, r; \quad t \in [0, 1]^d,
\]

where \( m_0 \) is a constant function (the \textit{grand mean}), \( \mu(t) \) is either zero or a non-constant function of \( t \) (the \textit{main effect} of \( t \)), \( a_i \) is either zero or a non-constant function of \( i \) (the \textit{main effect} of \( i \)) and \( \gamma_i(t) \) is either zero or a non-zero function which cannot be decomposed as a sum of a function of \( i \) and a function of \( t \) (the \textit{interaction} component).
IDENTIFIABILITY CONSTRAINTS

\[ \int_{[0,1]^d} \mu(t) \, dt = 0, \quad \sum_{i=1}^{r} a_i = 0, \]

\[ \sum_{i=1}^{r} \gamma_i(t) = 0, \quad \int_{[0,1]^d} \gamma_i(t) \, dt = 0, \quad \forall i = 1, \ldots, r; \quad t \in [0, 1]^d. \]
Difficulties of Pointwise ANOVA

“Dissipation of power.”

Fan & Lin (1998) proposed a powerful overall test for functional hypothesis testing → decomposition of the original functional data into Fourier (or wavelet) series expansions + adaptive Neyman and wavelet thresholding procedures of Fan (1996) to the resulting empirical Fourier (wavelet) coefficients.

Idea: *Sparsity* of data in non-standard (wavelet) domains.


- Optimality of Tests?

- Not discussed in FANOVA context.

- We derive asymptotically (as $\epsilon \to 0$ or, equivalently, as $n \to \infty$) optimal (minimax) non-adaptive and adaptive testing procedures for testing the significance of the main effect and the interactions in the FANOVA model against composite nonparametric alternatives (separated away from null in $L^2([0,1]^d)$-norm)

Hypotheses to be Tested 1

Testing the significance of the main effects and the interactions

\[ H_0 : \mu(t) \equiv 0, \quad t \in [0, 1]^d, \]
\[ H_0 : \gamma_i(t) \equiv 0, \quad \forall i = 1, \ldots, r, \quad t \in [0, 1]^d. \]

Identifiability constraints →

\[ Y_{i}^* = m_0 + a_i + \epsilon \xi_i, \quad i = 1, \ldots, r, \quad \sum_{i=1}^{r} a_i = 0, \]

where \( Y_{i}^* = \int_{[0,1]^d} dY_i(t) \) and \( \xi_i \) are independent \( \mathcal{N}(0, 1) \) random variables. This is the classical one-way fixed-effects ANOVA model.
Hypotheses to be Tested

We assume that \( m_i \) (and, hence, \( \mu \) and \( \gamma_i \) as well) belong to a Besov ball of radius \( C > 0 \) on \([0, 1]^d\), \( B^{s}_{p,q}(C) \), where \( s > 0 \) and \( 1 \leq p, q \leq \infty \).

Interested in: Rate at which the distance between the null and alternative hypotheses decreases to zero, while still permitting consistent testing. Alternatives are separated away from the null by \( \rho \) in the \( L^2([0, 1]^d) \).

Alternatives are of the form

\[
H_1 : \mu \in \mathcal{F}(\rho), \\
H_1 : \gamma_i \in \mathcal{F}(\rho), \quad \text{at least for one} \quad i = 1, \ldots, r,
\]

where \( \mathcal{F}(\rho) = \{ f \in B^{s}_{p,q}(C) : \|f\|_2 \geq \rho \} \).
Consider the general model

\[ dZ(t) = f(t) \, dt + \epsilon \, dW(t), \quad t \in [0, 1]^d, \]

where \( W \) is a \( d \)-dimensional standard Wiener process.

We wish to test

\[ H_0 : f \equiv 0 \quad \text{versus} \quad H_1 : f \in \mathcal{F}(\rho), \]

where \( \mathcal{F}(\rho) = \{ f \in B_{p,q}^s(C) : \| f \|_2 \geq \rho \} \).

For prescribed \( \alpha \) and \( \beta \), the rate of decay to zero of the “indifference threshold” \( \rho = \rho(\epsilon) \), as \( \epsilon \to 0 \), can be viewed as a measure of goodness of a test. It is natural to seek the test with the optimal (fastest) rate.
Minimax Optimality 2

[Ingster, 1993; Spokoiny, 1996; Ingster & Suslina, 2000].

**Definition** \(\rho(\epsilon)\) is the minimax rate of testing if \(\rho(\epsilon) \to 0\) as \(\epsilon \to 0\) and the following two conditions hold

(i) for any \(\rho'(\epsilon)\) satisfying \(\rho'(\epsilon)/\rho(\epsilon) = o_\epsilon(1)\), one has

\[
\inf_{\phi_\epsilon} \left[ \alpha(\phi_\epsilon) + \beta(\phi_\epsilon, \rho'(\epsilon)) \right] = 1 - o_\epsilon(1),
\]

where \(o_\epsilon(1) \to 0\) as \(\epsilon \to 0\).

(ii) for any \(\alpha > 0\) and \(\beta > 0\) there exists a constant \(c > 0\) and a test \(\phi^*_\epsilon\) such that

\[
\alpha(\phi^*_\epsilon) \leq \alpha + o_\epsilon(1), \quad \beta(\phi^*_\epsilon, c\rho(\epsilon)) \leq \beta + o_\epsilon(1).
\]

\(\phi^*_\epsilon\) is called an asymptotically optimal (minimax) test.
Minimax Optimality 3

Ingster (1993) and Lepski & Spokoiny (1999) showed that for \( sp > d \) the asymptotically optimal (minimax) rate is

\[
\rho(\epsilon) = \epsilon^{4s''/(4s''+d)},
\]

where \( s'' = \min(s, s - \frac{d}{2p} + \frac{d}{4}) \).

The proposed asymptotically optimal (minimax) tests were consistent but non-adaptive [involve the smoothness parameters \( s \) and \( p \) of the corresponding Besov ball].

Spokoiny (1996) and Horowitz & Spokoiny (2001): Problem of adaptive minimax testing where \( s \) and \( p \) are unknown. No adaptive test can achieve the exact optimal rate uniformly over all \( s \) and \( p \) (in some given range).
**Minimax Optimality 4**

- Price for Adaptivity: If one allows increase of \( \rho(\epsilon) \) by an additional log-log factor \( t_\epsilon = (\ln \ln \epsilon^{-2})^{1/4} \), i.e., considers \( \rho(\epsilon t_\epsilon) \) instead of \( \rho(\epsilon) \), then [Horowitz & Spokoiny (2001)] the optimal rate of adaptive testing is

  \[
  \rho(\epsilon t_\epsilon) = (\epsilon t_\epsilon)^{4s''/(4s''+d)},
  \]

- The “price” factor \( t_\epsilon \) is unavoidable and cannot be reduced.
Wavelet Bases

We assume $d = 1$ and work with periodic o.n. wavelet bases in $L^2([0,1])$ generated by shifts of a compactly supported scaling function $\phi$, i.e.

$$
\phi^p(t) = \sum_{\ell \in \mathbb{Z}} \phi(t-\ell), \quad \psi_{jk}^p(t) = \sum_{\ell \in \mathbb{Z}} \psi_{jk}(t-\ell), \quad j \geq 0, \quad k = 0, \ldots, 2^j - 1
$$

where

$$
\psi_{jk}(t) = 2^{j/2} \psi(2^j t - k).
$$

$\{\phi^p; \psi_{jk}^p, \quad j \geq 0, \quad k = 0, 1, \ldots, 2^j - 1\}$ generates an o.n. basis in $L^2([0,1])$.

If the MRA is of regularity $r > 0$, the corresponding wavelet basis is unconditional for Besov spaces $B^s_{p,q}([0,1])$ for $0 < s < r$, $1 \leq p, q \leq \infty$. Such bases characterize Besov balls in terms of wavelet coefficients.
■ Testing in FANOVA 1

Averaging over $r$ paths + identifiability conditions:

\[
\begin{align*}
    d\bar{Y}(t) &= (m_0 + \mu(t)) \, dt + \epsilon \, d\bar{W}(t), \quad t \in [0, 1] \\
    d(Y_i - \bar{Y})(t) &= (a_i + \gamma_i(t)) \, dt + \epsilon \, d(W_i - \bar{W})(t), \quad i = 1, \ldots, r.
\end{align*}
\]

\{W_i - \bar{W}; \ i = 1, \ldots, r\} are Wiener processes with the same covariance kernel $C(s, t) = \frac{r-1}{r} \min(s, t)$ [but no independent].

\[
\begin{align*}
    dZ(t) &= f(t) \, dt + \eta \, dW(t), \quad t \in [0, 1], \\
    Z(t) &= \bar{Y}(t), \quad f(t) = m_0 + \mu(t), \quad \eta = \epsilon/\sqrt{r} \\
    Z(t) &= (Y_i - \bar{Y})(t), \quad f(t) = a_i + \gamma_i(t), \quad \eta = \epsilon\sqrt{(r - 1)/r}
\end{align*}
\]
Testing in FANOVA 2

To apply Spokoiny (1996) results, assume that $B_{p,q}^s(C)$ satisfies $1 \leq p, q \leq \infty$, $sp > 1$ and $s - \frac{1}{2p} + \frac{1}{4} > 0$. [Donoho et al., 1995; Donoho & Johnstone, 1998].

$$H_0 : f \equiv \text{constant} \quad \left( = \int_0^1 f(t)dt \right)$$

versus

$$H_1 : \left( f - \int_0^1 f(t)dt \right) \in \mathcal{F}(\rho),$$

where $\mathcal{F}(\rho) = \{ f \in B_{p,q}^s(C) : \| f \|_2 \geq \rho \}$, $1 \leq p, q \leq \infty$, $sp > 1$ and $s - \frac{1}{2p} + \frac{1}{4} > 0$. 
Testing in FANOVA 3

Choose a wavelet $\psi$ of regularity $r > s$. One has

$$Y_{jk} = \theta_{jk} + \eta \xi_{jk}, \quad j \geq -1; \quad k = 0, 1, \ldots, 2^j - 1,$$

where $Y_{jk} = \int_0^1 \psi_{jk}(t)dZ(t)$, $\theta_{jk} = \int_0^1 \psi_{jk}(t)f(t)dt$ and $\xi_{jk}$ are independent $\mathcal{N}(0, 1)$ random variables.

Testing

$$H_0: f \equiv \text{constant}$$

is equivalent to testing

$$H_0: \theta_{jk} = 0 \quad \forall \ j \geq 0; \quad k = 0, 1, \ldots, 2^j - 1.$$
**NON-ADAPTIVE TEST 1**

**RESULT** Let the MRA be of regularity \( r > s \), and let the parameters \( s, p, q \) and the radius \( C \) of the Besov ball \( B^s_{p,q}(C) \) be known, where \( 1 \leq p, q \leq \infty \), \( sp > 1 \), \( s - \frac{1}{2p} + \frac{1}{4} > 0 \) and \( C > 0 \). Then, for a fixed significance level \( \alpha \in (0, 1) \), the test \( \phi^* \), for testing

\[
H_0 : f \equiv \text{constant} \quad \text{vs} \quad H_1 : \left( f - \int_0^1 f(t) dt \right) \in \mathcal{F}(\rho),
\]

where \( \mathcal{F}(\rho) = \{ f \in B^s_{p,q}(C) : \|f\|_2 \geq \rho \} \), is \( \alpha \)-level asymptotically optimal (minimax) test, as \( \eta \to 0 \). That is, for any \( \beta \in (0, 1) \), it attains the optimal rate of testing

\[
\rho(\eta) = \eta^{4s''/(4s'' + 1)},
\]

where \( s'' = \min\{s, s - \frac{1}{2p} + \frac{1}{4}\} \).
**NON-ADAPTIVE TEST 2**

- $\phi^*$ is based on the sum of squares of the thresholded empirical wavelet coefficients $Y_{jk}$ with properly chosen level-dependent thresholds. The null hypothesis is rejected when this sum of squares exceeds some critical value.

- $j_\eta$ the largest integer: $j_\eta \leq \log_2 \eta^{-2}$.

- $j(s)$ resolution level given by

$$j(s) = \frac{2}{4s'' + 1} \log_2 (C\eta^{-2}).$$

- Levels split as:

$$\mathcal{J}_- = \{0, \ldots, j(s) - 1\}, \quad \mathcal{J}_+ = \{j(s), \ldots, j_\eta - 1\}.$$
NON-ADAPTIVE TEST 3

For each $j \in J_-$, define

$$S_j = \sum_{k=0}^{2^j-1} (Y_{jk}^2 - \eta^2)$$

For each $j \in J_+$ and for given threshold $\lambda > 0$, define

$$S_j(\lambda) = \sum_{k=0}^{2^j-1} [(Y_{jk}^2 \mathbf{1}(|Y_{jk}| > \eta \lambda) - \eta^2 b(\lambda)],$$

where $b(\lambda) = \mathbb{E} [\xi^2 \mathbf{1}(|\xi| > \lambda)]$ and $\xi \sim \mathcal{N}(0, 1)$.
Define

\[ T(j(s)) = \sum_{j=0}^{j(s)-1} S_j, \]

and

\[ Q(j(s)) = \sum_{j=j(s)}^{j\eta-1} S_j(\lambda_j), \]

where \( \lambda_j = 4\sqrt{(j - j(s) + 8) \ln 2} \).

Under \( H_0 \),

\[ v_0^2(j(s)) = 2\eta^4 2^{j(s)} \quad \text{and} \quad w_0^2(j(s)) = \eta^4 \sum_{j=j(s)}^{j\eta-1} 2^j d(\lambda_j), \]

are the variances of \( T(j(s)) \) and \( Q(j(s)) \), respectively, where

\[ d(\lambda_j) = \mathbb{E} [\xi^4 1(\|\xi\| > \lambda_j)]. \]
NON-ADAPTIVE TEST 5

Comment: In MATLAB simulations we replaced the expression from Fan (1996):

\[
\mathbb{E}(\xi^{2k} 1(|\xi| > \lambda_j)) = \sqrt{\frac{2}{\pi}} \lambda_j^{2k-1} 2^{-8(j-j(s)+8)} + O \left( \lambda_j^{2k-3} 2^{-8(j-j(s)+8)} \right), \quad k = 1, 2, \ldots.
\]

by

\[
d(\lambda_j) = 3 - \sqrt{\frac{2}{\pi}} \Lambda_j^5 / 5 + \Lambda_j^7 / (7\sqrt{2\pi}) + o(\Lambda_j^8),
\]

where \( \Lambda_j = \min(\lambda_j, 1/\lambda_j) \). Similar approximation can be derived for \( b(\lambda_j) = \mathbb{E}(\xi^2 1(|\xi| > \lambda_j)) \).
Finally, for a given significance level $\alpha \in (0, 1)$, let $\phi^*$ be the test defined by

$$
\phi^* = \begin{cases} 
1 \{ T(j(s)) > v_0(j(s))z_{1-\alpha} \}, & \text{if } p \geq 2 \\
1 \left\{ T(j(s)) + Q(j(s)) > \sqrt{v_0^2(j(s)) + w_0^2(j(s))}z_{1-\alpha} \right\}, & \text{if } 1 \leq p < 2,
\end{cases}
$$
ADAPTIVE TEST 1

The parameters $s$, $p$, $q$ and the radius $C$ of the corresponding Besov ball $B^s_{p,q}(C)$ are unknown. Assume that $0 < s \leq s_{\text{max}}$, $1 \leq p, q \leq \infty$, $sp > 1$, $s - \frac{1}{2p} + \frac{1}{4} > 0$ and $0 < C \leq C_{\text{max}}$.

Let $t_\eta = (\ln \ln \eta^{-2})^{1/4}$ and $j_{\text{min}} = \frac{2}{4s_{\text{max}}+1} \log_2 \eta^{-2}$.

Regularity of MRA: $r > s_{\text{max}}$.

The idea: Consider the range of $j(s) = j_{\text{min}}, \ldots, j_\eta - 1$ and reject $H_0$ if it is rejected at least for one selected level $j(s)$. 
ADAPTIVE TEST 2

Since \( \text{card}(\{j_{\min}, \ldots, j_{\eta} - 1\}) = O(\ln \eta^{-2}) \), Bonferroni type testing leads to the asymptotically adaptive test

\[
\phi^{*}_{\eta} = 1 \left[ \max_{j_{\min} \leq j(s) \leq j_{\eta} - 1} \left\{ \frac{T(j(s)) + Q(j(s))}{\sqrt{v_{0}^{2}(j(s)) + w_{0}^{2}(j(s))}} \right\} > \sqrt{2 \ln \ln \eta^{-2}} \right].
\]
Spokoiny (1996) showed that the test $\phi^*_\eta$ is an adaptive optimal test, i.e.

$$\alpha(\phi^*_\eta) = o_\eta(1)$$

and

$$\sup_T \beta(\phi^*_\eta, c\rho(\eta t_\eta)) = o_\eta(1),$$

where $\rho(\eta t_\eta) = (\eta t_\eta)^{4s''/(4s''+1)}$, $o_\eta(1) \to 0$ as $\eta \to 0$, and $c$ is a constant.

If it is known that $p \geq 2$ then the adaptive test can be simplified to

$$\phi^*_\eta = 1 \left[ \max_{\min \leq j(s) \leq j_\eta - 1} \left\{ \frac{T(j(s))}{\sqrt{v_0^2(j(s))}} \right\} > \sqrt{2 \ln \ln \eta^{-2}} \right].$$
A COMMENT

The test $\phi^*_\eta$ is similar in spirit to that in Fan (1996) and Fan & Lin (2000), though they apply a \textit{global} threshold.
APPLICATIONS

SIMULATION STUDY 1

Synthetic data from the battery of standard test functions of Donoho & Johnstone (1995): BLOCKS, BUMPS, DOPPLER and HEAVISINE. Additional test function MISHMASH, defined as

\[ MISHMASH = -(\text{BLOCKS} + \text{BUMPS} + \text{DOPPLER} + \text{HEAVISINE}) \],

added because of the identifiability constraints.
Figure 1: The mean function $\mu(t) = 5\sin(2\pi t)$ and the centered treatment effect functions $\gamma_i(t)$, $i = 1, \ldots, 5$ (i.e., centered BLOCKS, BUMPS, DOPPLER, HEAVISINE, and MISHMASH), sampled at $n = 1024$ data points.
**SIMULATION STUDY 3**

- \( m_0 = 1, \mu(t) = 5 \sin(2\pi t) \)
- Five simulated observations (one for each test function shown; length \( n = 1024 \), two SNRs (SNR = 3 and 7)).
Figure 2: Five simulated observations (one for each test function shown in Figure 1) sampled at $n = 1024$ data points are shown superimposed (first plot) and separately (remaining five plots) for (a) SNR = 3 and (b) SNR = 7.
SIMULATION STUDY 4

To test the hypothesis $H_0 : \mu(t) = 0$, nonadaptive test, $p \geq 2$. Symmlet 8-tap

- $j(s) = 3$
- SNR=3: $T(3) = 15.28$ critical value 1.5949
- SNR=7: $T(3) = 97.52$ critical value 1.6316.

$H_0 : \gamma_i(t) = 0$ ($i = 1, \ldots, 5$), non-adaptive test, $1 \leq p < 2$. Daubechies 6-tap

- $j(s) = 3$  $j_\eta = 7.$
- SNR=3, $T(3) + Q(3) = 275.3326$ critical value 154.6294
- SNR=7, $T(3) + Q(3) = 5941.099$ critical value 156.4943
SIMULATION STUDY 5

Extensive power analysis for the above tests against the composite alternatives

\[ H_1 : \mu \in \mathcal{F}(\rho) \quad \text{and} \quad H_1 : \frac{1}{5} \sum_{i=1}^{5} \gamma_i \in \mathcal{F}(\rho). \]  \hspace{1cm} (1)
Figure 3: Empirical power functions for testing (a) $H_0 : \mu(t) = 0$ versus $H_1 : ||\mu||_2 = \rho$ and (b) $H_0 : \gamma_i(t) = 0$ ($i = 1, \ldots, 5$) versus $H_1 : ||\sum_i \gamma_i/5||_2 = \rho$. In both panels, the sample size was $n = 512$ and the number of trials at a fixed discretized SNR was 500.
ORTHOSIS DATA ANALYSIS 1

Interesting data on human movement.

Data: Amarantini David and Martin Luc, Laboratoire Sport et Performance Motrice, Grenoble University

Underlying movement under various levels of an externally applied force to the knee.

Seven young male volunteers wore a spring-loaded orthosis of adjustable stiffness under 4 experimental conditions:

- Control condition (without orthosis),
- Orthosis condition,
- Two conditions (Spring1, Spring2) stepping in place was perturbed by fitting a spring-loaded orthosis onto the right knee.
ORTHOSIS DATA ANALYSIS 2

The data set consists in 280 separate runs and involves the seven subjects over four described experimental conditions, replicated ten times for each subject.
Figure 4: Orthosis data set: panels in rows correspond to *Treatments* while the panels in columns correspond to *Subjects*.
ORTHOSIS DATA ANALYSIS 3: MODEL

Model

\[ dY_{ijk}(t) = m_{ij}(t) \, dt + \epsilon \, dW_{ijk}(t), \]

\[ i = 1, \ldots, I; \quad j = 1, \ldots, J; \quad k = 1, \ldots, K; \quad t \in [0, 1], \]

with

\[ m_{ij}(t) = m_0 + \mu(t) + \alpha_i + \gamma_i(t) + \beta_j + \delta_j(t), \]

\[ i = 1, \ldots, I; \quad j = 1, \ldots, J; \quad t \in [0, 1], \]

where \( i \) is the condition index, \( j \) is the subject index, \( k \) is the replication index, and \( t \) is the time.

Subjects in the above model are naturally considered as block effects; subjects obviously differ but the researchers are not interested in their differences.
orthosis data analysis 4: model

\[ d\bar{Y}_i..(t) = m_i(t) \, dt + \eta \, dW_i..(t), \ i = 1, \ldots, I; \ t \in [0, 1], \]

with

\[ m_i(t) = m_0 + \mu(t) + \alpha_i + \gamma_i(t), \ i = 1, \ldots, I; \ t \in [0, 1], \]

where \( \eta = \epsilon / \sqrt{JK} \).

- \( j(s) = 4 \) and \( j_\eta = 6 \).
- Coiflet 18-tap filter
ORTHOSIS DATA ANALYSIS 5

Tests $H_0 : \mu(t) = 0$ and $H_0 : \gamma_i = 0$ were both significant.

The researchers interested contrasts:

- Control and Orthosis functional treatment effects are equal ($H_0 : \gamma_1(t) = \gamma_2(t)$). Not significant, $p$-value 0.157.
- Spring 1 and Spring 2 functional treatment effects are equal ($H_0 : \gamma_3(t) = \gamma_4(t)$). Not significant, $p$-value 0.198.
- Contrast $(\gamma_1(t) + \gamma_2(t)) - (\gamma_3(t) + \gamma_4(t))$. Significant, $p$-value is 0.0103.
Figure 5: Empirical estimators of the treatment effects of interest. Constant and functional components $\alpha_i$ and $\gamma_i(t)$ ($i = 1, \ldots, 4$) are not separated.
**CONCLUSIONS**

- \( dY(s, t) = \)
- \((m_0 + a(s) + \mu(t) + \gamma(s, t)) \, dt \, ds + \epsilon \, dW(s, t), \quad (s, t) \in [0, 1]^2 \)
- \( d \geq 2, \) [Thresholding? Block Thresholding, FDR?]
- Black Box Procedure: Variances of \( T, S \) by bootstrap [wavestrap, Percival et al. 1999].
- Data, Matlab Files: brani@isye.gatech.edu.