Links Between Binary and Multi-Category Logit Item Response Models and Quasi-Symmetric Loglinear Models

Alan Agresti
Department of Statistics
University of Florida
Gainesville, Florida 32611-8545

July 18, 2006

Written for *Annales de la Faculté des Sciences de l'Université de Toulouse* in honor of the retirement of Professor Henri Caussinus from Université Paul Sabatier in Toulouse, France

Abstract

Caussinus's loglinear model of quasi symmetry has interesting connections with models for within-subject effects with repeated categorical measurement. For binary responses, Tjur (1982) showed that estimates of main effect parameters in the quasi-symmetry model are also conditional maximum likelihood estimates of item parameters for a fixed effects treatment of subject terms in the Rasch item response model. He showed they are also nonparametric estimates of item parameters for a random effects treatment of subject terms in the Rasch model. I describe some generalizations of the quasi-symmetry model that have similar connections with generalizations of the Rasch model. These include a link between an ordinal quasi-symmetry model and an adjacent-categories logit model with random effects, and a link between a multivariate quasi-symmetry model and a logit random effects model for repeated measurement of a multivariate vector of binary responses.

Keywords: Adjacent-categories logit; Conditional maximum likelihood; Cumulative logit; Ordinal; Random effects; Rasch model.

1 Introduction

The Caussinus (1966) quasi-symmetry model is one of the most useful models for analyzing contingency tables having the same categories for each classification. Like many, I became aware of this model and its utility by the detailed discussion of it and its generalizations in the seminal text on loglinear models by Bishop, Fienberg, and Holland (1975). In my study of categorical data methods in the period 1975-1990, I became increasingly aware of its connections with other standard models, such as the Bradley-Terry model for paired evaluations (Fienberg and Larntz 1976).

In the past ten years, some of my own research has dealt with extensions of this model as well as connections between it and certain logit models for repeated measurement having subject-specific terms. This paper summarizes these research results. The logit models of interest are extensions of the Rasch model. One of them is a generalization to multivariate binary responses. Two others refer to ordinal generalizations. The three standard types of ordinal logit models are (1) cumulative logit models, which use all cumulative probabilities and their complements, (2) adjacent-categories logit models, which use all pairs of probabilities from adjacent categories, and (3) continuation-ratio logits (sometimes also called "sequential logits"), which use each category probability together with the probability of a lower response, or each category probability together with the probability of a higher response. We consider (1) and (2); see Tutz (1990) for (3).

I begin by reviewing the Rasch model. Suppose n subjects respond to T items (e.g., questions on an exam or questionnaire) that use the same c categories. For subject i and item t, let Y_{it} denote the response outcome. For the binary-response case ($c = 2$), the Rasch model is

$$\text{logit}[P(Y_{it} = 1)] = \alpha_i + \beta_t, \quad i = 1, ..., n, \ t = 1, ..., T. \tag{1}$$

(This and all other models in this paper require constraints for identifiability. For simplicity of exposition, we will not discuss these.) The usual assumption for model fitting is local independence for the repeated responses by a subject, given the subject effect.

Rasch treated subject parameters $\{\alpha_i\}$ as fixed effects, but much subsequent work treats them as random effects. Tjur (1982) studied a distribution-free approach for them. He showed that the marginal distribution, integrating out the random effects, satisfies a multiplicative model. Although he did not note it, that model is in fact the quasi-symmetry model, as pointed out in related work of the same era (Fienberg 1981, Fienberg and Meyer 1983). Since similar results occur for models discussed in this paper, the next section outlines an argument that connects the Rasch and quasi-symmetry models.

2 Quasi Symmetry and the Rasch Model

Cross-classifying responses on the T binary items yields a 2^T contingency table. Let $\mathbf{Y}_i = (Y_{i1}, \ldots, Y_{iT})$ denote the sequence of T responses for subject i, which contributes an observation to a particular cell of this table. For a possible sequence of outcomes $\mathbf{r} = (r_1, ..., r_T)$ for $\mathbf{Y}_i$, where each $r_t = 1$ or 0,

$$P(\mathbf{Y}_i = \mathbf{r}|\alpha_i) = \prod_t \left[\frac{\exp(\alpha_i + \beta_t)}{1 + \exp(\alpha_i + \beta_t)}\right]^{r_t} \left[\frac{1}{1 + \exp(\alpha_i + \beta_t)}\right]^{1-r_t}$$

$$= \frac{\exp[\alpha_i(\sum_t r_t) + \sum_t r_t\beta_t]}{\prod_t[1 + \exp(\alpha_i + \beta_t)]}.$$

With a random effects approach, let F denote the cumulative distribution function of α_i. Then the marginal probability of sequence $\mathbf{r}$ for the responses $\mathbf{Y}$ for a randomly selected subject is (suppressing the subject label)

$$P(\mathbf{Y} = r) = \exp(\sum_t r_t\beta_t) \int \frac{\exp[\alpha(\sum_t r_t)]}{\prod_t[1 + \exp(\alpha + \beta_t)]} dF(\alpha).$$

This probability contributes to the likelihood, which is that for a multinomial distribution over the 2^T cells for possible $\mathbf{r}$. Regardless of the choice for F, the integral is complex. However, it depends on the data only through $s = \sum_t r_t$, so a more general model replaces this integral by a separate parameter for each value of this sum. This model has form

$$\log P(\mathbf{Y} = \mathbf{r}) = \sum_t r_t \beta_t + \lambda_s. \tag{2}$$

The term λ_s in the implied marginal model (2) represents an interaction parameter $\lambda_{r_1,\ldots,r_T}$ that is the same at each value of $s = \sum_t r_t$. These interaction parameters result from the marginal dependence in responses, due to heterogeneity in $\{\alpha_i\}$. The interaction term is invariant to any permutation of the response outcomes $(r_1, \ldots, r_T)$, since each such permutation yields the same sum. Because of this symmetry in interaction, it is the extension of Caussinus's loglinear model of quasi symmetry for the T-way table in the binary response case (Bishop et al. 1975).

No matter what form the random effects distribution takes, the implied marginal model has the same main effects structure, and it has an interaction term that is a special case of the one in (2). Thus, one can consistently estimate the item effects $\{\beta_t\}$ using the ordinary ML estimates for the quasi symmetry model. In fact, Tjur (1982) showed that these estimates are also the conditional ML estimates of $\{\beta_t\}$ for model (1), treating $\{\alpha_i\}$ as fixed effects and conditioning on their sufficient statistics.

Tjur (1982) also proved that these quasi-symmetric ML estimators and conditional ML estimators for the Rasch model are identical to those obtained in a slightly extended version of ML for a nonparametric treatment of the distribution of α_i. Later papers showed strong connections between the actual nonparametric marginal ML estimates and conditional ML estimates. Under the assumption that the Rasch model holds, de Leeuw and Verhelst (1986) showed that the probability that nonparametric ML estimators are identical to conditional ML estimators (and hence also to quasi-symmetric loglinear ML estimators) converges to 1 as n increases, for a fixed number of items. Lindsay *et al.*

(1991) strengthened this, showing the same result if the subject-effect distribution has at least $(T+1)/2$ support points.

Darroch (1981), Fienberg (1981), Kelderman (1984), and Hatzinger (1989) made related observations about the connection between the Rasch and quasi-symmetry models. I found similar connections useful in research on modeling rater agreement (Agresti and Lang 1993a) and capture-recapture modeling for estimating population size (Agresti 1994), as did others for these and related applications (Darroch and McCloud 1986, Becker 1990, Darroch et al. 1993, Fienberg et al. 1999).

An extension of the Rasch form of model for nominal response variables is

$$\log[P(Y_{it} = j)/P(Y_{it} = c)] = \alpha_{ij} + \beta_{tj},\ j = 1, ..., c-1. \tag{3}$$

Similar connections with quasi symmetry occur for this model. The conditional ML estimates of the item effects are identical to estimates of main effect parameters in the general quasi-symmetry loglinear model for a c^T contingency table (Conaway 1989, McCullagh 1982). For expected frequencies $\{\mu_{ab...c}\}$ in that table, the quasi-symmetry model has form

$$\log \mu_{ab...c} = \lambda_{a1} + \lambda_{b2} + ... + \lambda_{cT} + \lambda_{ab...c}, \tag{4}$$

where the interaction term is symmetric in its indices.

3 Quasi Symmetry and an Ordinal Model Using Adjacent-Category Logits

I considered extensions of Tjur's results for ordinal responses and for multivariate binary responses. First consider an ordinal model that has the adjacent-categories logit representation for the response for subject i on item t,

$$\log[P(Y_{it} = j+1)/P(Y_{it} = j)] = \alpha_{ij} + \beta_t. \tag{5}$$

This is a special case of the nominal-scale model in which the item effects have the structure $\beta_{t,j+1} - \beta_{tj} = \beta_t$ for all j; that is, $\{\beta_{tj}\}$ are linear in j. The item effects are assumed to be identical for each pair of adjacent categories. A somewhat simpler model decomposes α_{ij} in (5) into $\alpha_i + \delta_j$ (Andersen 1973, Andrich 1978, Duncan 1984, Hout *et al.* 1987, Agresti 1993a).

Agresti (1993a) showed that conditional ML estimates and extended nonparametric marginal ML estimates of the item effects in model (5) are identical to the ordinary ML estimates obtained in fitting the loglinear model

$$\log \mu_{ab\ldots c} = a\beta_1 + b\beta_2 + \cdots + c\beta_T + \lambda_{ab\ldots c} \ , \tag{6}$$

where λ is permutationally invariant. This is a special case of the quasi-symmetry model that has linear structure for the main effects. It treats the main effects as variates, with equally-spaced scores, rather than qualitative factors. Each main effect term has a single parameter, rather than the $c-1$ parameters in the Caussinus model. Model (6) is an *ordinal quasi-symmetry* model, since it reflects the ordering of the response categories. Agresti (1993a) also showed that estimates of $\{\beta_t\}$ for the model with simpler structure for α_{ij} equal those for a simpler loglinear model in which the interaction parameter depends only on the sum of the scores for the T items. For examples of the use of model (6), see Agresti (1993a, 1993b, 1995). It is simple to fit the model using software for generalized linear models. See Agresti (1996, p. 277) for the use of SAS (PROC GENMOD).

The ML estimates of $\{\beta_t\}$ in (6) have the same order as the sample mean responses (using equally-spaced scores) in the T one-way margins of the c^T table, as those are the sufficient statistics for $\{\beta_t\}$. The complete symmetry model for a c^T contingency table is the special case of (6) in which $\beta_1 = \ldots = \beta_T$. Given that model (6) holds, marginal homogeneity is equivalent to symmetry. When model (6) fits well, one can test marginal homogeneity using a likelihood-ratio test with $df = T - 1$, based on comparing its fit to that of complete symmetry. This is an ordinal analog of the Caussinus (1966) test of marginal homogeneity based on comparing the ordinary quasi-symmetry model to the complete symmetry model.

4 Quasi Symmetry and an Ordinal Model Using Cumulative Logits

An alternative model form for ordinal responses uses cumulative logits. For subject i and item t, the cumulative logit analog of model (5) is

$$\log[P(Y_{it} \leq j)/(1 - P(Y_{it} \leq j))] = \alpha_{ij} - \beta_t. \tag{7}$$

This model has the proportional odds property, for which the item effects $\{\beta_t\}$ are identical at each j. Complete symmetry is implied by $\beta_1 = ... = \beta_T$.

The conditional ML approach does not apply to model (7) since these logits are not the canonical parameters for the multinomial. Agresti and Lang (1993b) eliminated the subject parameters by noting that (7) corresponds to a Rasch model for all $c - 1$ binary collapsings of the response, with the same item effects for each collapsing. Hence, because of the connection between Rasch models and quasi symmetry, one can estimate the item parameters by fitting a quasi-symmetry model simultaneously to all such binary collapsings, using the same main effect parameters for each. They did this using methods for maximizing a likelihood subject to constraints (Lang and Agresti 1994).

See Agresti and Lang (1993b) and Agresti (1993b, 1995) for examples and a more detailed discussion of this approach. Samejima (1969), Andrich (1978), Masters (1982), Duncan (1984), and Tutz (1990) described related models for ordinal responses. Hedeker and Gibbons (1994) presented a random effects approach for a simpler form of the subject term.

5 Quasi Symmetry and Analyses of Ordinal Matched Pairs

This section considers separately the matched-pairs case $T = 2$ with an ordinal response. In this case, quasi-symmetry models have simple logit representations, and additional ways exist of obtaining item estimates. These models refer to probabilities $\{\pi_{ab}\}$ for the $c \times c$ table of counts $\{n_{ab}\}$ for the pairs of possible responses for the n subjects.

Section 3 noted that the logit model (5) for adjacent categories relates to a special ordinal version (6) of quasi symmetry. Letting $\beta = \beta_2 - \beta_1$ in that loglinear model, it is equivalent to the logit model (Agresti 1983),

$$\log(\pi_{ab}/\pi_{ba}) = \beta(b - a). \tag{8}$$

This is a special case of Goodman's diagonals-parameter symmetry model, with a linear trend for the diagonals parameters (Goodman 1979; for related material, see Goodman 2002). One can also estimate β using software for logistic regression models, treating $\{n_{ab},\ a < b\}$ as independent binomial variates with sample sizes $\{n_{ab} + n_{ba}\}$.

Simple ordinal tests of marginal homogeneity derive from model (8). A Wald test uses as test statistic the ratio of $\hat{\beta}$ to its asymptotic standard error. The likelihood-ratio test compares this model with the symmetry model. Rao's efficient score test is based on the difference in sample means for the marginal distributions, for equally-spaced category scores. Specifically, let $\{p_{ab}\}$ denote the sample proportions in the observed $c \times c$ table. A z test statistic is the ratio of $d = [\sum_a a(p_{a+} - p_{+a})]$ to its estimated standard error, which is the square root of $(1/n)[\sum_a \sum_b (a - b)^2 p_{ab} - d^2]$.

For cumulative logit model (7) with $T = 2$, a simple estimate of $\beta = \beta_2 - \beta_1$ uses the fact that the model implies a Rasch model for each of the $c - 1$ collapsings of the response to a binary variable. For each collapsing, the off-main-diagonal cells of the

2×2 table provide an estimate in the form of the binary conditional ML estimate for two items, $\log(n_{12}/n_{21})$. A nearly efficient estimator results by combining these $c-1$ estimates, adding the numerators and adding the denominators before taking their ratio and their logarithm (Agresti and Lang 1993b). In terms of the cell counts $\{n_{ab}\}$ in the full $c \times c$ table, the resulting estimate is

$$\tilde{\beta} = \log\{[\sum_{a<b}(b-a)n_{ab}]/[\sum_{a>b}(a-b)n_{ab}]\}. \tag{9}$$

The estimated asymptotic variance of this estimator equals

$$\hat{V}(\tilde{\beta}) = \frac{\sum_{a<b}(b-a)^2 n_{ab}}{[\sum_{a<b}(b-a)n_{ab}]^2} + \frac{\sum_{a>b}(a-b)^2 n_{ab}}{[\sum_{a>b}(a-b)n_{ab}]^2}.$$

Another simple test of marginal homogeneity for ordinal matched-pairs data uses $z = \tilde{\beta}/\sqrt{\hat{V}(\tilde{\beta})}$. Like the test based on the ordinal quasi-symmetry model, it is sensitive to location shifts in the marginal distributions. McCullagh (1977) discussed other estimators for the cumulative logit model applied to matched pairs.

6 Quasi Symmetry and a Multivariate Logit Model for Repeated Measurement

A multivariate extension of the Rasch model also has connections to quasi-symmetric loglinear models. It refers to V separate binary variables, each measured for T items. For subject i, denote the response for item t with variable v by Y_{itv}, with observed value 1 or 0. Consider the model

$$\text{logit}[P(Y_{itv} = 1)] = \alpha_{iv} + \beta_{tv}. \tag{10}$$

For each variable v, this model has the additive subject and item form of the Rasch model. The $\{\beta_{1v}, ..., \beta_{Tv}\}$ for each v describe the item effects for each variable. The $\{\alpha_{iv}\}$ reflect the heterogeneity among subjects that induces the correlations among repeated responses on a variable.

Agresti (1997) gave a nonparametric treatment of $\boldsymbol{\alpha}_i = (\alpha_{i1}, ..., \alpha_{iV})$, treating this as a vector of correlated random effects. Integrating out the random effects yields a marginal model for the outcomes $\mathbf{y} = (y_{11}, \ldots, y_{TV})$ on the TV combinations of items and variables with expected frequencies $\{\mu_{\mathbf{y}}\}$ in a 2^{TV} contingency table. Regardless of the joint distribution for those random effects, this model satisfies

$$\log \mu_{\mathbf{y}} = \sum_t \sum_v \beta_{tv} y_{tv} + \lambda(\sum_t y_{t1}, ..., \sum_t y_{tV}), \tag{11}$$

where the final term represents a separate parameter for each possible ordered set of the V sums of item scores. Specifically, model (10) implies that a marginal model has the same main effects structure as (10), and it has an interaction term that is a special case of the one in (11). Thus, one can consistently estimate $\{\beta_{tv}\}$ in a nonparametric manner using the ordinary ML estimates for the loglinear model. Moreover, the conditional ML estimates of $\{\beta_{tv}\}$ for model (10) are identical to the ordinary ML estimates of $\{\beta_{tv}\}$ obtained by fitting loglinear model (11).

For this loglinear model, the interaction involving any set of items for a particular variable has term that is invariant for any permutation of the response outcomes for those items. For the univariate case, model (11) is the quasi-symmetry model. Thus, model (11) is a *multivariate quasi-symmetry model*.

In the matched-pairs case ($T = 2$), model (11) has fitted values in the 2×2 marginal table for each variable that are identical to the observed counts. The estimate of $\exp(\beta_{2v} - \beta_{1v})$ then equals the number of cases with $(y_{1v}, y_{2v}) = (0, 1)$ divided by the number of cases with $(y_{1v}, y_{2v}) = (1, 0)$. In the univariate case ($V = 1$), this is also the conditional ML estimate for the logit model, and Neuhaus et al. (1994) showed that it is also normally the estimate for a parametric random effects approach.

7 Summary

This paper has discussed the connection between item response models and quasi-symmetric loglinear models. Other articles that dealt in part with this connection or exploited it to fit an item response form of model with loglinear software include Fienberg and Meyer (1983), Kelderman (1984), Fischer et al. (1986), Kelderman and Rijkes (1994), and Erosheva et al. (2002). Ten Have and Becker (1995) discussed a wide variety of loglinear models with quasi-symmetric structure

One can extend the models of this article to incorporate covariates, as long as the main focus is on within-subjects effects. For instance, one might stratify a sample by some group factor (e.g., gender), and analyze whether the same item effects apply for each group. One could do this by comparing the fits of two models, one assuming homogeneous item effects and the other permitting heterogeneous item effects. The related quasi-symmetry models also have homogeneous or heterogeneous main effects, with the symmetric interaction term having different parameters for each group. Agresti (1993b) gave examples of this type.

In my experience, quasi-symmetry models very often fit quite well, even for large sample sizes. This may partly reflect the fact that the Rasch form of model is a natural one for many applications. Moreover, quasi-symmetry models address components of relationships not analyzed by standard loglinear analyses. When quasi-symmetry models show lack of fit, they usually still fit much better than complete symmetry or mutual independence loglinear models. From their structure of heterogeneous main effects and their connection with Rasch-like models, ordinal quasi-symmetry models are designed to detect shifts in location among margins of the c^T table. Thus, they may fit poorly when marginal distributions show differences in dispersion as well as location.

In summary, the quasi-symmetry model benefits from wide scope, from close connections with other useful models, and from ease of generalization to other models for multinomial or multivariate repeated categorical responses. The statistical community as well as methodologists who frequently deal with categorical responses owe Professor Henri Caussinus their grateful thanks and congratulations for introducing this model.

Acknowledgments

This work was partially supported by grants in the U.S. from the National Institutes of Health and the National Science Foundation. Thanks to Prof. Stephen Fienberg for helpful comments.

References

Agresti, A. (1983). A simple diagonals-parameter symmetry and quasisymmetry model. *Statistics and Probability Letters*, **1**, 313-316.

Agresti, A. (1993a). Computing conditional maximum likelihood estimates for generalized Rasch models using simple loglinear models with diagonals parameters. *Scandinavian Journal of Statistics*, **20**, 63-72.

Agresti, A. (1993b). Distribution-free fitting of logit models with random effects for repeated categorical responses. *Statistics in Medicine*, **12**, 1969-1987.

Agresti, A. (1994). Simple capture-recapture models permitting unequal catchability and variable sampling effort. *Biometrics*, **50**, 494-500.

Agresti, A. (1995). Logit models and related quasi-symmetric loglinear models for comparing responses to similar items in a survey. *Sociological Methods in Research*, 68-95.

Agresti, A. (1996). *An Introduction to Categorical Data Analysis.* New York: Wiley.

Agresti, A. (1997). A model for repeated measurements of a multivariate binary response, *Journal of the American Statistical Association*, **92**, 315-321.

Agresti, A., and J. B. Lang. (1993a). Quasi-symmetric latent class models, with application to rater agreement. *Biometrics*, **49**, 131-139.

Agresti, A., and J. B. Lang. (1993b). A proportional odds model with subject-specific effects for repeated ordered categorical responses. *Biometrika*, **80**, 527-534.

Andersen, E. B. (1973). Conditional inference for multiple-choice questionnaires. *Brit. J. Math. Statist. Psychol.*, **26**, 31-44.

Andrich, D. (1978). A rating formulation for ordered response categories. *Psychometrika*, **43**, 561-573.

Becker, M. P. (1990). Quasisymmetric models for the analysis of square contingency tables. *Journal of the Royal Statistical Society, Series B*, **52**, 369-378.

Bishop, Y. M. M., S. E. Fienberg, and P. Holland. (1975). *Discrete Multivariate Analysis.* MIT Press.

Caussinus, H. (1966). Contribution a l'analyse statistique des tableaux de correlation. *Annales de la Faculté des Sciences de l'Université de Toulouse.* **29**, 77-182.

Conaway, M. (1989). Analysis of repeated categorical measurements with conditional likelihood methods. *Journal of the American Statistical Association*, **84**, 53-62.

Darroch, J. N. (1981). The Mantel-Haenszel test and tests of marginal symmetry; fixed effects and mixed models for a categorical response. *International Statistical Review*, **49**, 285-307.

Darroch, J. N., and P. I. McCloud. (1986). Category distinguishability and observer agreement. *The Australian Journal of Statistics*, **28**, 371-388.

Darroch, J. N., S. E. Fienberg, G. F. V. Glonek, and B. W. Junker. (1993). A three-sample multiple-recapture approach to census population estimation with heterogeneous catchability. *Journal of the American Statistical Association*, **88**, 1137-1148.

De Leeuw, J., and N. Verhelst. (1986). Maximum likelihood estimation in generalized Rasch models. *Journal of Educational Statistics*, **11**, 183-196.

Duncan, O. D. (1984). Rasch measurement: Further examples and discussion. Pp. 367-403 in *Surveying Subjective Phenomena*, vol. 1, edited by C. F. Turner and E. Martin. New York: Russell Sage Foundatiom.

Erosheva, E., S. E. Fienberg, and B. Junker. (2002). Alternative statistical models and representations for large sparse multi-dimensional contingency tables. *Annales de la Faculté des Sciences de l'Université de Toulouse.*

Fienberg, S. E. (1981). Recent advances in theory and methods for the analysis of categorical data: Making the link to statistical practice. *Bull. Int. Statist. Inst.*, **49**, Book 2, 763-791.

Fienberg, S. E., and M. M. Meyer. (1983). Loglinear models and categorical data analysis with psychometric and econometric applications. *Journal of Econometrics*, **22**, 191-214.

Fienberg, S. E., and K. Larntz. (1976). Loglinear representation for paired and multiple comparison models. *Biometrika*, **63**, 245-254.

Fienberg, S. E., M. Johnson, and B. Junker. (1999). Classical multi-level and Bayesian approaches to population size estimation using data from multiple lists. *J. Roy. Statist. Soc. (A)*, **162**, 383-406.

Fischer, G. W., M. S. Kamlet, S. E. Fienberg, and D. Schkade. (1986). Risk preferences for gains and losses in multiple objective decisionmaking. *Management Science*, **32**, 1065-1086).

Goodman, L. A. (1979). Multiplicative models for square contingency tables with ordered categories. *Biometrika*, **66**, 413-418.

Goodman, L. A. (2002). Contributions to the statistical analysis of contingency tables: Notes on quasi-symmetry, quasi-independence, log-linear models, log-bilinear models, and correspondence analysis models. *Annales de la Faculté des Sciences de l'Université de Toulouse.*

Hatzinger, R. (1989). The Rasch model, some extensions and their relation to the class of generalized linear models. *Statistical Modelling: Proceedings of GLIM89 and the 4th International Workshop on Statistical Modelling.* Lecture Notes in Statistics, **57**, Berlin: Springer.

Hedeker, D., and R. D. Gibbons. (1994). A random-effects ordinal regression model for multilevel analysis. *Biometrics*, **50**, 933-944.

Hout, M., O. D. Duncan, and M. E. Sobel. (1987). Association and heterogeneity: Structural models of similarities and differences. *Sociological Methodology*, **17**, 145-184.

Kelderman, H. (1984). Loglinear Rasch model tests. *Psychometrika*, **49**, 223-245.

Kelderman, H., and C. P. M. Rijkes. (1994). Loglinear multidimensional IRT models for polytomously scored items. *Psychometrika*, **59**, 149-176.

Lang, J. B., and A. Agresti. (1994). Simultaneously modeling joint and marginal distributions of multivariate categorical responses. *Journal of the American Statistical Association*, **89**, 625-632.

Lindsay, B., C. C. Clogg, and J. Grego. (1991). Semiparametric estimation in the Rasch model and related exponential response models, including a simple latent class model for item analysis. *Journal of the American Statistical Association*, **86**, 96-107.

Masters, G. N. (1982). A Rasch model for partial credit scoring. *Psychometrika*, **47**, 149-174.

McCullagh, P. (1977). A logistic model for paired comparisons with ordered categorical data. *Biometrika*, **64**, 449-453.

McCullagh, P. (1982). Some applications of quasisymmetry. *Biometrika*, **69**, 303-308.

Neuhaus, J. M., J. D. Kalbfleisch, and W. W. Hauck. (1994). Conditions for consistent estimation in mixed-effects models for binary matched-pairs data. *Canadian Journal of Statistics*, **22**, 139-148.

Rasch, G. (1961). On general laws and the meaning of measurement in psychology. pp. 321-333 in *Proc. 4th Berkeley Symp. Math. Statist. Probab.*, vol. 4, ed. J. Neyman. Berkeley: Univ. of California Press.

Samejima, F. (1969). Estimation of latent ability using a response pattern of graded scores. *Psychometrika, Monograph Supplement*, **17**.

Ten Have, T. R., and M. P. Becker (1995). Multivariate contingency tables and the analysis of exchangeability. *Biometrics*, **51**, 1001-1016.

Tjur, T. (1982). A connection between Rasch's item analysis model and a multiplicative Poisson model. *Scandinavian Journal of Statistics*, **9**, 23-30.

Tutz, G. (1990). Sequential item response models with an ordered response. *British Journal of Mathematical and Statistical Psychology*, **43**, 39-55.